What About Other Central Forces

There are various possibilities for the central force law. How do we know that the hodoyne represents the true motion of planets in a priori fashion? Why is the central force law of $F \propto \frac{1}{R^2}$ the only a priori possibility for planets? One thing we know is that the hodograph of a repetitive orbit must be consistent with a planet that comes around 360 degrees in orientation. Logic tells us that the velocity vectors must therefore progress smoothly; from one vector to the next, the direction in which they point must rotate in a continuous curve of some kind to complete a 360 degree rotation. In a sense, they must "come around". For the hodoyne in *Orbits Explained*, the vectors were shown to rotate around so that if they were all placed with their tails at a point, then an exact circle would be traced by their tips. But there could be other possibilities as we shall see. Bear in mind that the force is always directed centrally, so any set of velocity vectors will have to be logically consistent with this fact.

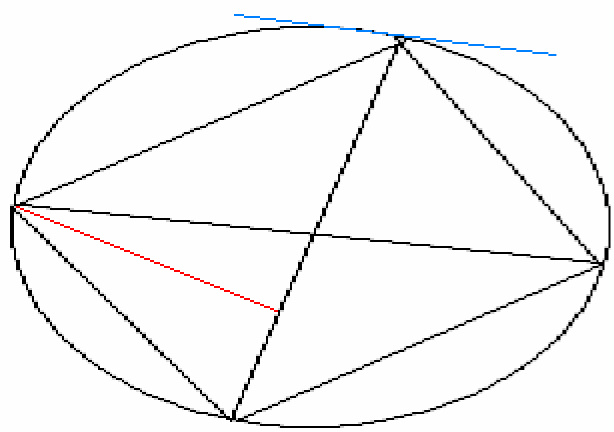
Plato used elegant logic to prove that the square root of two is irrational. He showed that it could not be an odd integer divided by an even integer, nor even divided by odd, nor odd divided by odd; thus he eliminated what he showed to be all the possibilities – an ingenious tactic. We will use the tactic of eliminating all possibilities, aside from an inverse square law of force, for planetary orbits. We will stay with a priori methods.

There are only three possibilities for central attractive force. The force can decrease as distance increases. The force can increase as distance increases. The force can stay the same as distance increases. We will see that the first possibility is the only valid one.

A century ago Joseph Louis Francois Bertrand proved that there are only two possible central force laws for repetitive orbits. One is the inverse square law of force and the other is Hooke's Law wherein centrally directed force increases with distance as with a mechanical spring. I can not

comprehend his proofs, lacking the mathematical talent. But we have obtained some mathematical tools and logic in *Orbits Explained* and they turn out to be enough to produce an a priori proof for Bertrand's findings.

We will however need a new tool that is given to us by Apollonius concerning a useful property of ellipses. He told us that all of the parallelograms formed by connecting the ends of any two conjugate diameters of an ellipse contain equal areas. A diameter of an ellipse is a line through the center of the ellipse. Now imagine a diameter of the ellipse where it touches the ellipse. Imagine the tangent line to the ellipse at that point. Then choose as the second diameter, the one that is parallel to the aforementioned tangent line. These two diameters are defined as conjugate diameters of the ellipse.

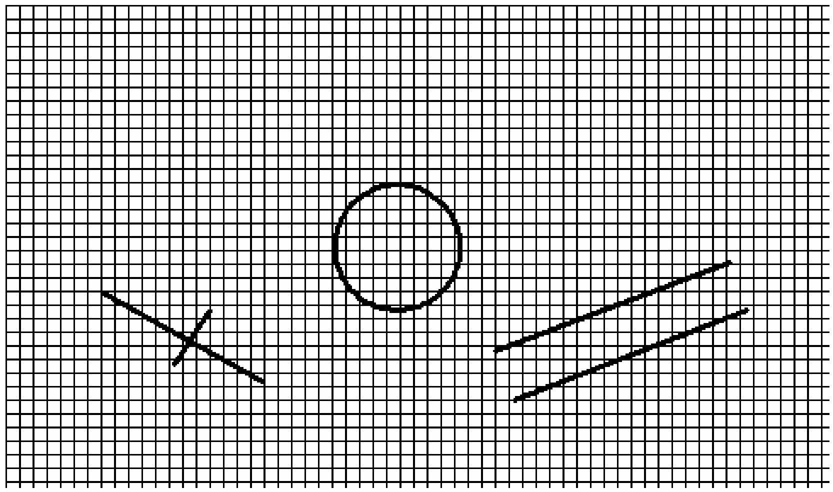


For the moment ignore the red line in the figure above. Note that the tangent at the end of one diameter is drawn in blue. Parallel to the blue tangent is the conjugate diameter.

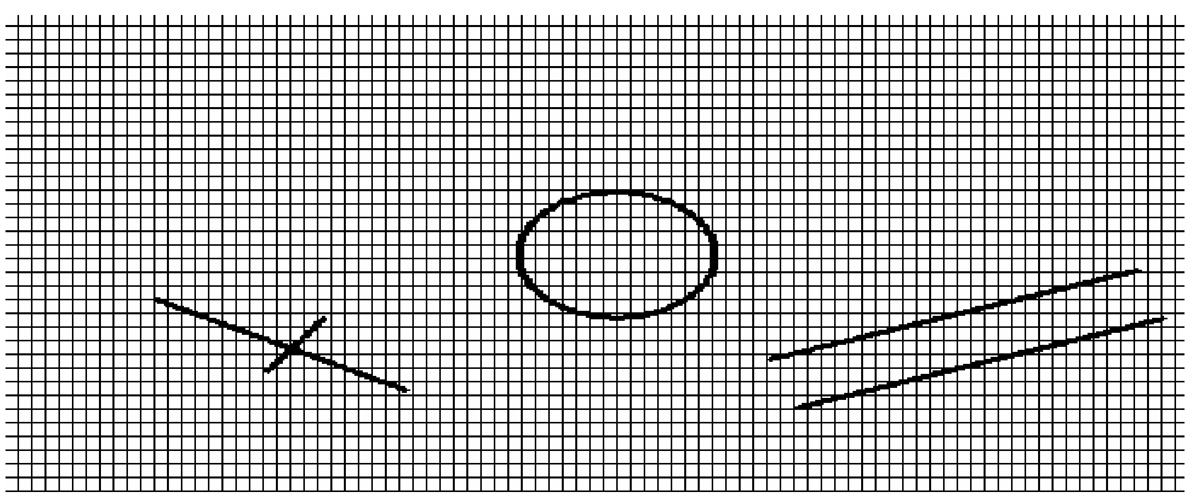
We will need a proof for Apollonius' Theorem which states that all sets of parallelograms containing conjugate diameters of the ellipse contain equal areas. It was difficult to find a direct proof for this, and I give credit for some insightful concepts on the mathematics website of The University of Crete. They use the concept of affine transformations which we will borrow.

One type of affine transformation is to stretch a given curve in one direction. Let's examine the typical Cartesian Coordinate system otherwise known as the "x y axis". We will prove that the affine transformation stretches a circle into an ellipse. We will also prove that parallel lines remain parallel and that proportions of lines retain their proportions after the affine transformation.

In the figure below there is a set of curves – a circle, a pair of parallel lines, and a set of intersecting bisecting lines. These curves are drawn on a grid which represents x and y coordinates.

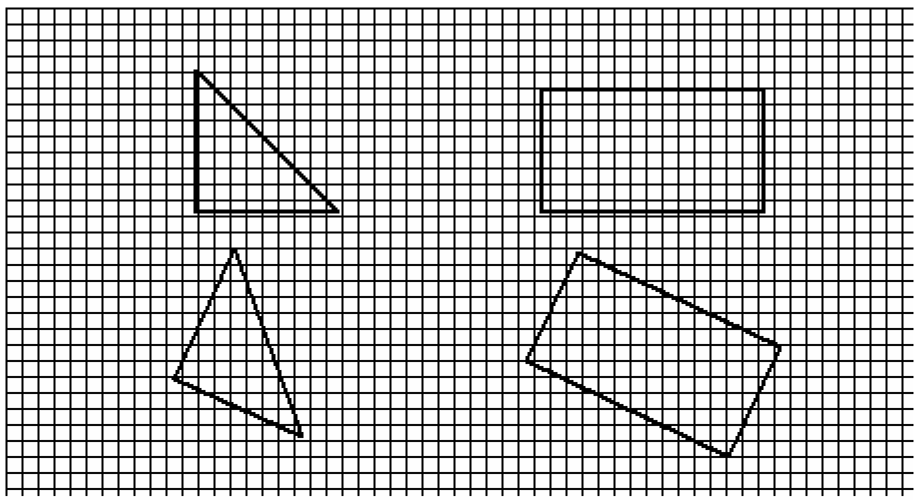


Each point along the curves has its x and y value. Suppose we want to stretch the curves horizontally. We would keep the y value unchanged and increase the value of x for each point. In the next figure we have increased the value of x by approximately 50%.

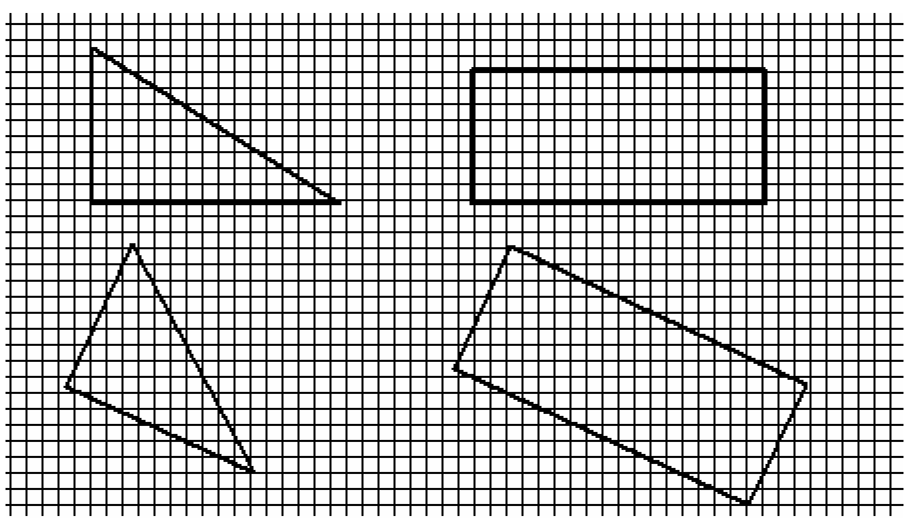


The circle has been transformed into an ellipse. The parallel lines are still parallel but their slopes and lengths have changed. The intersecting lines still bisect each other but their slopes and length have changed.

The next concept is a little tricky to demonstrate. For any given shape of area, regardless of its orientation, the area increases by the same amount when stretched horizontally. In the figure below, let the areas of the two triangles be equal to each other and let the areas of the two rectangles be equal to each other.



Now stretch the areas horizontally:



The rectangles will contain areas equal to each other after being stretched horizontally even though they are tilted relative to each other. The same can be said for the two triangles. A little thought tells us why this is true. Suppose that instead of stretching the x axis like we have done, we instead simply change the scale of the x axis. For example the height of each box in the grid still remains one unit of y but the horizontal side of each little box is worth 2 units of x. We would not need to stretch the areas horizontally. The effect of transforming the scale of the horizontal x axis is effectively the same as the effect of stretching the shapes horizontally while leaving the scale unchanged. It is clear that if we rotate a shape of given area on a graph where the scale of the horizontal x axis is changed, the literal area of the shape does not change because we did not have to stretch it. The calculated area, when we take into account the change in scale of the x axis, will be a new amount of area but all the resultant stretched shapes will share that same new amount of area. Thus we know that in the horizontal affine transformation, proportions of areas do not change. A square tilted in various orientations will stretch into parallelograms at various orientations all of which have areas equal to each other.

Let's prove that the circle transforms into an ellipse. The standard equation of a unit circle is $x^2 + y^2 = 1$. Solve for y:

$$y = \sqrt{1 - x^2}$$

Now modify the circle so that for every point on the circle the curve is stretched horizontally. This means the x value is reduced by a constant factor which we will call k. Y now becomes:

$$y = \sqrt{1 - (kx)^2}$$

And

$$y^2 = 1 - k^2 x^2$$

So:

$$y^2 + k^2 x^2 = 1$$

Divide through by k squared:

$$\frac{y^2}{k^2} + \frac{x^2}{1} = 1$$

And we recognize this to be the equation for an ellipse.

Now let's examine the parallel lines to see that they remain parallel after the horizontal affine transformation. The equation of a line is

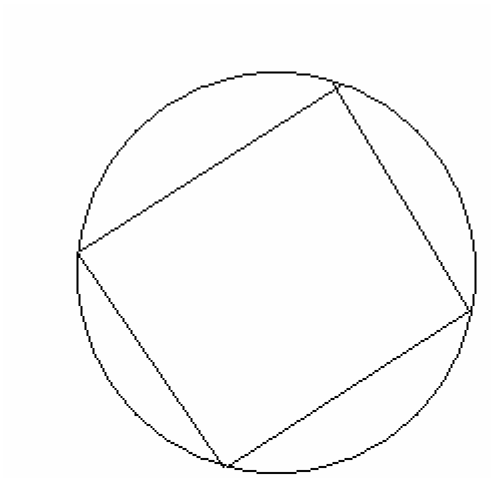
$y = mx + b$ where m is the slope of the line. If we modify the scale of x and keep y constant we are effectively modifying the term m to the same degree for all lines and so parallel lines will remain constant after stretching horizontally.

Now let's examine intersecting bisecting lines. By inspection of they are stretched as the value of x increases by an equal proportion for every point on the line, the proportions of the lines are preserved. This bisecting lines will still bisect each other after they are stretched horizontally.

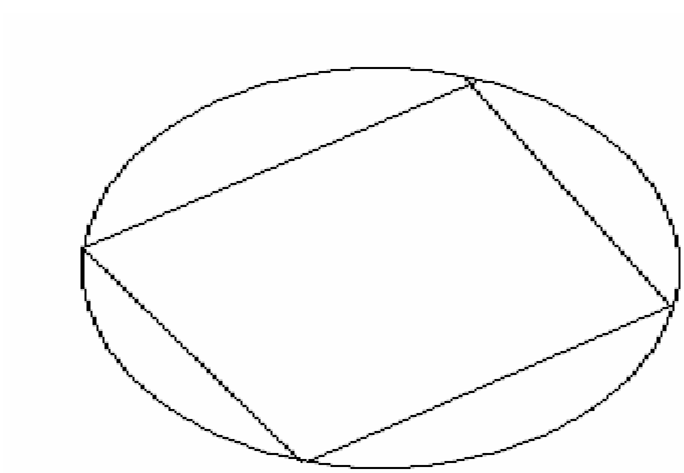
So we see the basic properties of the horizontal affine transformation that we will be using. A circle transforms into an ellipse. Parallel lines stay parallel. Bisecting lines still bisect each other.

Now we have enough knowledge to prove Apollonius Theorem that all the parallelograms inscribed in a given ellipse have areas equal to each other.

A square can be drawn inscribed in a circle at any orientation:



After the horizontal transformation all the parallelograms inscribed in the ellipse will have areas equal to each other since we showed that proportions of areas are preserved after the transformation. For the above orientation, the affine transformation results are shown below:

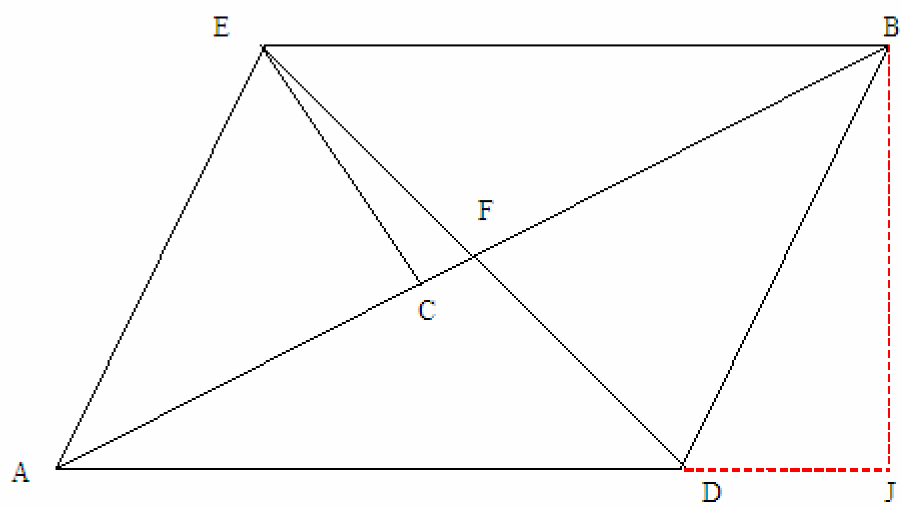


We know that for any orientation of the square in the circle, the areas of the parallelograms in the ellipse after the transformation will have equal areas to each other and so we have a proof for Apollonius' Theorem.

Next we will examine a way to measure the area of parallelogram. We have a standard method which uses the base times the height. But a less popular method using the diameter suits our purposes better.

In the figure below, for the parallelogram AEBD, we could use the standard area formula,

$area = AD \times BJ$



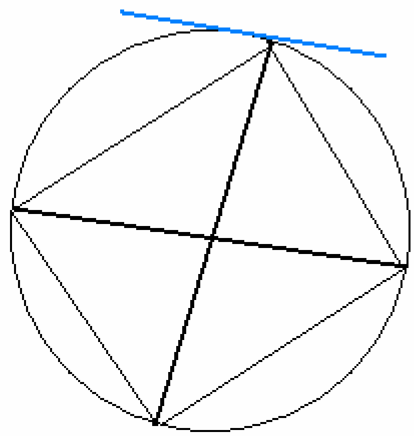
But there is another way to find the area of AEBD. Let angle ECF be a right angle created when a perpendicular to the diameter is drawn to the corner of the parallelogram. By inspection, the area of the top two triangles, ACE and CEB sum to half the area of the parallelogram. We write the formulae for the top two triangles:

$$areaAEB = \frac{AC \times CE}{2} + \frac{CB \times CE}{2} = \frac{AC + CB}{2}(CE) = \frac{AB \times CE}{2}$$

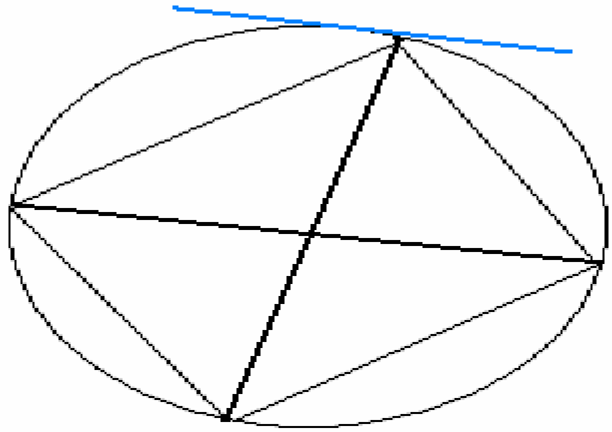
So we see that the area of triangle AEB is the length of the diameter of the parallelogram times the length of the perpendicular CE. Area AEB is half the parallelogram area by inspection so the total area of the parallelogram is twice AEB and is therefore equal to $AB \times CE$. In words the area of the parallelogram is equal to the diameter times its perpendicular to its corner.

We will put this to good use in combination with the affine transformation to prove Apollonius' Theorem which states that the areas of parallelograms whose diameters are conjugate diameters of a given ellipse are all equal.

The horizontal affine transformation shows us that if we start with an obliquely oriented square circumscribed in a circle and the tangent to one of its diameters, the tangent will still be parallel to the conjugate diameter after the transformation.

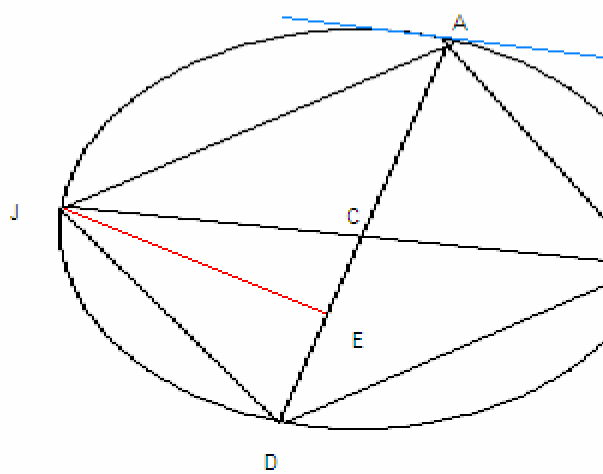


Note in the figure above that the blue tangent is parallel to the square's other diameter before the transformation. We showed that parallel lines stay parallel after the transformation so the blue tangent will still be parallel to what becomes by definition a conjugate diameter in the ellipse:



We actually do not need to use this application of the affine transformation since we can simply define a conjugate diameter and use Apollonius Theorem of constant area of parallelograms inscribed in ellipses to proceed with our proofs. But the affine transformation can be seen to produce conjugate diameters and thus is presented here for interest.

It is time to put our new tools together to build an elliptical orbit. In this case we will see that the central attractive body will be at the center of the ellipse instead of at a focus. Note in the figure below that an ellipse is drawn with a pair of conjugate diameters. The tangent to one of the diameters is drawn in blue.



The perpendicular EJ from the diameter DA is drawn to the corner of the parallelogram. The area of inscribed parallelograms in an ellipse is constant. So as we showed above, $EJ \times DA$ is equal to the area and is a constant for all inscribed parallelograms. Bisected lines remain bisected after the horizontal transformation so CA is one half of the diameter DA. Since $EJ \times DA$ is constant, so is $EJ \times CA$. This means that EJ is inversely proportional to CA. We can assign CA to represent radius to the planet as measured from the center of the ellipse and EJ to represent the tangential velocity since radius and tangential velocity are inversely proportional to each other according to the a priori proof that equal areas are swept in equal times. Then by using the vector component method of projection presented in *Orbits Explained* we see that the total velocity of the planet is CJ since it is parallel to the tangent of the ellipse, the tangent giving the direction of total velocity. As the planet travels along the ellipse at the tip of its radius CA, the velocity vectors CJ trace an ellipse at their tips. Thus the velocity hodograph for a central attractive body at the center of an ellipse is itself an ellipse.

By inspection equal areas are swept as the planet travels each quadrant of the ellipse since the quadrants are all of equal area. Thus the planet travels each quadrant in equal times. Note that the radius from the center of the ellipse goes from minimum to maximum in one quadrant and from maximum to minimum in the next quadrant and again from minimum to maximum and so on. If we stand at the center of the ellipse and view the radius to the planet as the planet orbits, we see the radius grow to a maximum in the same time it takes to decrease to a minimum. The only active force is towards the center of the ellipse since the planet has no thrust of its own. We recognize this oscillation of radius from maximum to minimum and minimum to maximum in equal times as simple harmonic motion according to Hooke's Law as it applies to mechanical springs. Therefore the central force obeys Hooke's Law and increases in direct proportion to the increase in radius. The farther away the planet is, the stronger the restoring force towards the center.

We have two possible orbits so far. The first is the orbit with the Sun at a focus of an ellipse and the force inversely proportional to the square of the distance. The second is the orbit with the Sun at the center of an ellipse and the force increasing directly with distance. There is a third possibility for the central force of an orbit by logic, that of force being constant, i.e. unrelated to distance. If force does not change with distance there must be a possible circular orbit for a Sun at the center of the circle. There could not be an elliptical orbit since if the planet were to increase its radius from the Sun, the force would not be able to increase so as to prevent the planet from escaping. There would be only one possible orbit for each given solar system. Perurbations from other masses no matter how far away would alter the planets velocity and cause the planet to escape so the possibility for stable arrangements of matter in the universe would be nil. Logic tells us that we can rule out the possibility that attractive force at a distance can be unchanging with distance.

Next we can rule out the possibility that force increases directly with distance according to Hooke's Law. If this were the case for the universe, then distant masses would be attracted to each other with tremendous force which would cause the universe to collapse. Logically this can not be the a priori law of force for the universe or the motion of the planets.

Of the three possibilities for central force, that it increases with distance, is unrelated to distance, or decreases with distance, only the latter is therefore logical and correct. The hodograph narrows the possibility range from all central forces which diminish with distance to the single possibility that the force diminishes with the square of distance. The velocity vectors must come around in orientation 360 degrees for an orbit. The force must always be directed centrally. The central force, diminishing with distance, can only logically result in a hodograph that is a circle or an ellipse. The velocity vectors' hodograph could not be of any other shape since a diminishing force with distance only allows the velocity vectors to rotate 360 degrees one time during a single orbit and smoothly at that. No other shapes exist that can comply with this. A "flower petal" or any other shape makes no sense since a concavity in the curve for the hodograph would spiral the planet into the Sun, terminating the repetitive orbit.. It would imply decreasing velocity not necessarily matched by an increased radius and therefore a spiral inward. As for the possibility that some other convex curve for the hodograph is a possibility, consider the following. In order for an orbit to repeat exactly, it is logical that there must be a symmetry of radial velocity with regard to directions that are 180 degrees from each other relative to the central attractive body. That is to say the planet must recede from the Sun along the radius to the Sun at an equal magnitude but opposite direction on opposite sides of the Sun. We saw this for the hododyne orbit with the Sun at a focus of the ellipse. It is also evident in the hodograph for the Sun at the center of the ellipse as in Hooke's Law of central force. Looked at in different perspective, there must be a net sum total radial velocity of zero per orbit, otherwise the orbit could not be repetitive. Radial velocity must change smoothly since the only force is directed centrally and the planet must progress from one point to the next without skipping positions on its orbit. It would be impossible to assemble a set of radial velocities which sum to zero if they are not pairs on exactly opposite sides of the Sun and still obey the rule that tangential velocity must be inversely proportional to radius for a convex velocity hodograph. The occurrence of a sporadic radial velocity , out of smooth sequence would produce a concavity in the velocity hodograph. Let's visualize in a priori fashion why radial velocity must be of equal magnitude on opposite sides of the Sun. There are only 2 possible arrangements, either the Sun is at the center of the orbit or it is off center. If it is off center such as at the focus of an ellipse, there are 2 symmetrical parts of the orbit, divided by the single major axis. If the Sun is at the center of the orbit such as at the center of a circle or ellipse, there are 4 symmetrical parts of the orbit divided by the intersecting major and minor axes. The Sun is the only determining factor for force on the planet and so there is no other way to orient the planet aside from the planet's position relative to the Sun. Careful thought affirms that each symmetrical part of the orbit must have an identical array of characteristics. The angle as measured from an axis, the radius, and the velocity must be identical for each symmetrical part. Otherwise, the orbit would not repeat. An orbit composed of symmetrical parts can not repeat unless the symmetrical parts are identical because the only orientation is relative to the Sun. It is as if the Sun does not care if the planet is in the East or West since there are no directions. The Sun only cares that the planet repeats its symmetrical trek exactly, through the various parts of the orbit, as the Sun views the planet from the Sun's own perspective and position. Trying various sketches of orbital curves such as circles, ellipses, egg shapes, and ovals, one is easily convinced that for each symmetrical part of an orbit there can only be one reversal of tendency for radial velocity. In other words, the radial velocity can increase and then decrease, but it can not then increase again before leaving that symmetrical part of the orbit. Otherwise, the planet would have to be subject to some force other than the centrally directed force of the Sun. This being the case, we are led to the suspicion that radial velocity on opposite sides of the Sun must be equal in magnitude. The symmetry suggests that when we assemble symmetrical parts of an orbit into an entire orbit, the radial velocities will be mirror images of each other on opposite sides of the Sun. An insight into this arrangement can be gained from the analysis of a ball thrown up into the air and then falling to the ground. We know from basic laws of motion that the radial velocity relative to the ground is exactly equal but in opposite direction for any position of the ball, when we compare its motion on the way up to the motion on the way down. Now, visualize what happens if we theoretically take away the physical Earth and therefore take away the ground but let the hypothetical center of force remain where the ground used to be. The ball will not crash. Instead, it will fall past the point of central force and travel a distance equal to the height it was thrown "up" and then it will return past the central point again. In other words it will oscillate up and down past the central point symmetrically. Its radial velocity will be a mirror image on either side of the central point. The next philosophical step is to allow a real situation to occur instead of a hypothetical one. Let a central attracting body, a Sun, to exist and imagine that we want a planet to be thrown up and around it, so that it oscillates like the ball did. In order for it to repeat its orbit, the radial velocity must be a mirror image for each symmetrical part of the orbit as it was for the ball. The only difference is that each "up and down" becomes each "symmetrical part and the next symmetrical part".

If we accept that radial velocity must be equal in magnitude on opposite sides of the Sun, and we acknowledge that equal areas are swept in equal times, and acknowledge that the tendency of radial velocity can only reverse in tendency once per symmetrical part of an orbit, and acknowledge that radial velocity must average to zero for an orbit, we can eliminate concave hodograph shapes for naturally occurring orbits. Furthermore we can be assured that only two hodographs are possible. One is elliptical and one is circular. Sketches and analysis would convince us that no other hodograph shapes could comply with the aforementioned factors that must be acknowledged. A simple way to convince oneself would be to start with one of the two hodographs that are valid and alter the curve of the hodograph into some shape other than a circle or an ellipse. One finds that at least one of the aforementioned required acknowledgements will not hold true. It is philosophically a bit like the method of mathematical induction whereby we can prove that some premise can or can not be valid when subjecting another known truth to an alteration, as per the premise to be tested, and then analyzing the effect to see if the initial truth is violated.

In conclusion, the elliptical orbit, with circular orbits being a special case, is the only possible shape for orbits around a central attractive body because no other curve can geometrically satisfy the symmetry of radial velocity and the inverse proportion between radius and tangential velocity while the total velocity is oriented along the tangent to the curve. As for the possible velocity hodograph shapes, we have eliminated the elliptical one as the rule for the universe since the universe would collapse. The circular one is the only satisfactory possibility. The central force law must be the inverse square law.

Perhaps we therefore see that the hododyne of *Orbits Explained* leads to an a priori proof for planetary laws. Force must diminish with the square of distance, a priori. Orbits are elliptical with the Sun at a focus, a priori.

With good intentions,
D.S.M.