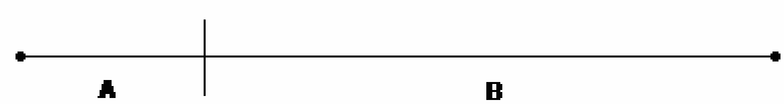
The Inverse Proportion Machine

If not for pencils and rulers, graphing paper and dinner napkins, this chapter would not exist. It is one of my principles that one should be mildly but not completely discrete about where and how to doodle in public. When an irresistible idea comes, it may become necessary to disappear quietly in order to find some writing materials and a private spot where the thing can be jotted down or the math can be put to the test. But then, if one is with family, perhaps all discretion is lost and the pencil can appear without the preliminary excuses. All these moments, in the aggregate, constitute an obsession. In this chapter I present the result of endless perseverating and scribbling, -the hododyne.

What is an inverse proportion?

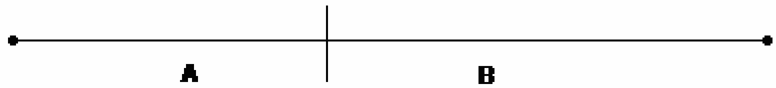
We are going to need a device that will govern the magnitude of two quantities so that the two quantities will be inversely proportional to each other. For now the two quantities could be anything that can be measured. It doesn’t matter yet what things we will actually measure with our device. All we want is a device or a machine that will always give us an inverse proportion of things. If we can have a machine like this, we will be able to rig it to comply with the motion of planets.

First, let me offer some words of caution. One must be exact in the definition inverse proportions. It is easy to be misled. Stating that one thing is inversely proportional to another does not simply mean that one thing gets smaller when the other gets larger. That is not enough of a description. For instance, cut a horizontal line of constant length into two parts.



Name the segment to the left of the cut A and the segment to the right of the cut B.

As the cut moves toward the right , A increases at the expense of B. But A and B are not inversely proportional.



A and B are not truly inversely proportional because the arrangement does not satisfy an additional requirement which is that the length of A multiplied by the length of B must always equal a constant result.

In terms of an equation, if A and B are inversely proportional, then A times B equals a constant.

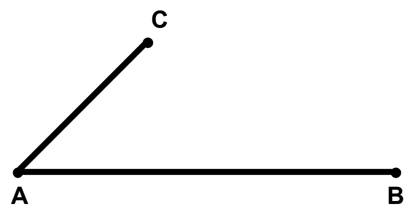
Equivalently, A equals a constant divided by B. Again equivalently, B equals a constant divided by A. This true inverse proportion can be exceedingly tricky to represent geometrically. But that feat will be accomplished by The Inverse Proportion Machine.

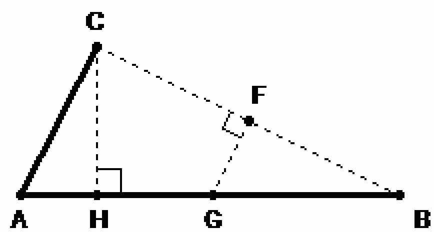
Introducing: The Inverse Proportion Machine

The Inverse Proportion Machine is a device that will generate two distances between points on a line that are truly inversely proportional. We start with a line $\overline{CAB}$

and make a bend in it at point A that is not at the midpoint of the line. Now there will be two connected segments.

Bend the segment farther:

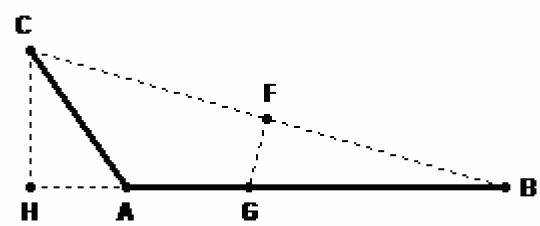




The longer segment $\overline{AB}$ we will call the base. Call the shorter segment $\overline{AC}$ the stick. As the stick rotates around point A where it meets the base the angle between the base and the stick changes.

For any angle a line $\overline{CB}$ can be drawn from the end of the stick to the far end of the base. We name this line the connector. We give the connector at its midpoint a point, F . From this point F draw a perpendicular bisector $\overline{FG}$ of the connector that hits the base at point G. Also draw a line $\overline{CH}$ that is perpendicular to the base and intersects the point C at the far end of the stick $\overline{AC}$.

When the angle between the segments exceeds 90 degrees, extend the base as in the figure below so that the perpendicular segment $\overline{CH}$ can reach up to the point C. Place point H on the dotted extension of the base.



Next, take the bisector of the connector and let it intersect with the base at point "G". Lastly label the far end of the base point "B".

<u>Inversely proportional segments</u>

It will be proven that for any angle between the base and the stick, the distance $\overline{AG}$ multiplied by the distance $\overline{HB}$ will be a constant. In other words, even as the angle between $\overline{AB}$ and $\overline{AC}$ changes the distance $\overline{HB}$ is truly inversely proportional to the distance $\overline{AG}$.

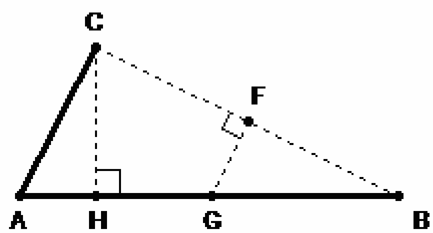
Specifically it will be shown that for any angle between the rotating legs of the Inverse Proportion Machine, the product $\overline{AG}$ multiplied by the distance $\overline{HB}$ will be equal to $\frac{\overline{AB}^2 - \overline{AC}^2}{2}$. Note that since the legs $\overline{AB}$ and $\overline{AC}$ are constant for any particular Inverse Proportion Machine, the expression $\frac{\overline{AB}^2 - \overline{AC}^2}{2}$ is also a constant.

<u>Less than 90 degrees</u>

Now for the proof of The Inverse Proportion Machine. The proof will be done separately for various arrangements of the two segments of the machine. The first proof is for the situation where the angle between the legs of the machine is less than 90 degrees. Then the proof is extended to include angles greater than 90 degrees, exactly 90 degrees, exactly 180 degrees and finally overlapping segements.

Thus all the possible arrangements of the segments will be covered by the proof.

The previous chapter demonstrated the Smaller Hypotenuse Property that is valid for any right triangle whose hypotenuse is bisected in perpendicular fashion. Notice such a right triangle ΔCHB within the figure below:



The Smaller Hypotenuse Property reveals that $\overline{CB}^2 = 2\overline{HB}\left(\overline{GB}\right)$

Before applying the Smaller Hypotenuse Property, notice that:

$\overline{AH}^2 + \overline{CH}^2 = \overline{AC}^2$ by Pythagorus.

$\overline{HB}^2 + \overline{CH}^2 = \overline{CB}^2$ by Pythagorus.

$\overline{AB} = \overline{AH} + \overline{HB}$ = constant since $\overline{AB}$ is a segment of fixed length of the Inverse Proportion Machine.

$\overline{AC}$ is also constant since it is the other fixed length segment of the machine.

$\overline{HB}^2 = \overline{CB}^2 - \overline{CH}^2$ and:

$\overline{CH}^2 = \overline{AC}^2 - \overline{AH}^2$ by rearrangement of Pythagorus above so:

Substituting for $\overline{CH}^2$:

$\overline{HB}^2 = \overline{CB}^2 - \left(\overline{AC}^2 - \overline{AH}^2\right) = \overline{CB}^2 + \overline{AH}^2 - \overline{AC}^2$

By the Smaller Hypotenuse Property of right triangles, as stated above $\overline{CB}^2 = 2\overline{HB}\left(\overline{GB}\right)$ so:

$\overline{HB}^2 = 2\overline{HB\left(\overline{GB}\right)} + \overline{AH}^2 - \overline{AC^2}$

By inspection of the machine $\overline{AH} = \left(\overline{AB} - \overline{HB}\right)$ so by substtution:

$$\overline{HB}^2 = 2\overline{HB}\left(\overline{GB}\right) + \left(\overline{AB} - \overline{HB}\right)^2 - \overline{AC}^2$$

Now examine the term $\left(\overline{AB} - \overline{HB}\right)^2$ in the equation above.

Multiply it out:

$$\left(\overline{AB} - \overline{HB}\right)^2 = \left(\overline{AB} - \overline{HB}\right)\left(\overline{AB} - \overline{HB}\right) = \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) + \overline{HB}^2$$

Substitute this expression into the equation for $\overline{HB}^2$.

$$\overline{HB}^2 = 2\overline{HB}\left(\overline{GB}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) + \overline{HB}^2 - \overline{AC}^2$$

Subtract $\overline{HB}^2$ from both sides:

$$0 = 2\overline{HB}\left(\overline{GB}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) - \overline{AC}^2$$

By inspection of the machine $\overline{GB} = \overline{AB} - \overline{AG}$ so by substitution:

$0 = 2\overline{HB}\left(\overline{AB} - \overline{AG}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) - \overline{AC}^2$ and by multiplying out:

$0 = 2\overline{AB}\left(\overline{HB}\right) - 2\overline{HB}\left(\overline{AG}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) - \overline{AC}^2$ so by arithmetic:

$0 = \overline{AB}^2 - 2\overline{HB}\left(\overline{AG}\right) - \overline{AC}^2$ and by rearrangement:

$\overline{AB}^2 - \overline{AC}^2 = 2\overline{HB}\left(\overline{AG}\right)$ and finally:

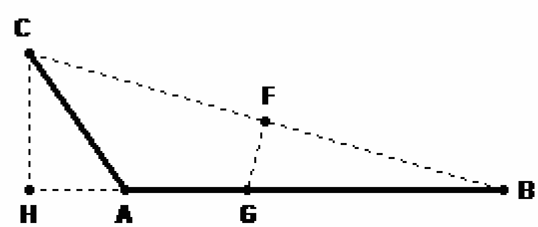
$$\frac{\overline{AB}^2 - \overline{AC}^2}{2} = \overline{HB}\left(\overline{AG}\right)$$

Notice the left side of the equation. Realize that the Inverse Proportion Machine is composed of two legs of constant length $\overline{AB}$ and $\overline{AC}$. That means that $\overline{AB}$ and $\overline{AC}$ are constants and that the left side of the equation is constant. This means that the product of the segments $\overline{HB}$ and $\overline{AG}$ is always constant.

Therefore according to the strict mathematical definition of inverse proportions, the lengths of the segments $\overline{HB}$ and $\overline{AG}$ are inversely proportional to each other. The Inverse Proportion Machine, for this configuration of the segments $\overline{AB}$ and $\overline{AC}$ whereby the angle between them is less than 90 degrees, is therefore valid. Next is the proof for the condition that the angle between the segments $\overline{AB}$ and $\overline{AC}$ is greater than 90 degrees.

<u>Greater than 90 degrees</u>

By inspection of the figure below see that the principal right triangles have slightly different arrangements compared to the case above.



$\overline{AH}^2 + \overline{CH}^2 = \overline{AC}^2$ by Pythagorus.

$\overline{HB}^2 + \overline{CH}^2 = \overline{CB}^2$

$\overline{AB} = \overline{HB} - \overline{AH}$. Notice the sign change from the proof above for the angle less than 90 degrees. Also note that $\overline{AB}$ is constant since it is a leg of the machine.

$$\overline{HB}^2 = \overline{CB}^2 - \overline{CH}^2$$

$\overline{CH}^2 = \overline{AC}^2 - \overline{AH}^2$ by rearrangement of the equation above so:

$$\overline{HB}^2 = \overline{CB}^2 - \left(\overline{AC}^2 - \overline{AH}^2\right) = \overline{CB}^2 + \overline{AH}^2 - \overline{AC}^2$$

By the Smaller Hypotenuse Property of right triangles stated above, $\overline{CB}^2 = 2\overline{HB}\left(\overline{GB}\right)$ is still true by inspection for the triangle ΔHBC above so:

Substituting for $\overline{CB}^2$:

$$\overline{HB}^2 = 2\overline{HB\left(\overline{GB}\right)} + \overline{AH}^2 - \overline{AC^2}$$

By inspection of the machine $\overline{AH} = \left(\overline{HB} - \overline{AB}\right)$ so:

$$\overline{HB}^2 = 2\overline{HB}(\overline{GB}) + (\overline{HB} - \overline{AB})^2 - \overline{AC}^2$$

It is always true algebraically that $(x - y)^2 = (y - x)^2$. For example $(5 - 3)^2 = (3 - 5)^2 = 4$.

So, note that $(\overline{HB} - \overline{AB})^2 = (\overline{AB} - \overline{HB})^2 = \overline{HB}^2 - 2\overline{HB}(\overline{AB}) + \overline{AB}^2$. This is a finding that makes the two proofs merge.

$$(\overline{HB} - \overline{AB})^2 = (\overline{HB} - \overline{AB})(\overline{HB} - \overline{AB}) = \overline{HB}^2 - 2\overline{AB}(\overline{HB}) + \overline{AB}^2$$

Substituting for $(\overline{HB} - \overline{AB})^2$:

$$\overline{HB}^2 = 2\overline{HB}(\overline{GB}) + \overline{HB}^2 - 2\overline{AB}(\overline{HB}) + \overline{AB}^2 - \overline{AC}^2$$

Subtract $\overline{HB}^2$ from both sides:

$$0 = 2\overline{HB}(\overline{GB}) + \overline{AB}^2 - 2\overline{AB}(\overline{HB}) - \overline{AC}^2$$

By inspection of the machine $\overline{GB} = \overline{AB} - \overline{AG}$ so substituting for $\overline{GB}$:

$$0 = 2\overline{HB}\left(\overline{AB} - \overline{AG}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) - \overline{AC}^2$$

$$0 = 2\overline{AB}\left(\overline{HB}\right) - 2\overline{HB}\left(\overline{AG}\right) + \overline{AB}^2 - 2\overline{AB}\left(\overline{HB}\right) - \overline{AC}^2$$ and by arithmetic:

$$0 = \overline{AB}^2 - 2\overline{HB}\left(\overline{AG}\right) - \overline{AC}^2$$

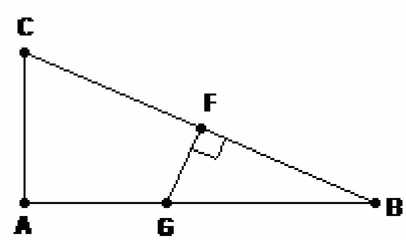
$$\overline{AB}^2 - \overline{AC}^2 = 2\overline{HB}\left(\overline{AG}\right)$$

$$\frac{\overline{AB}^2 - \overline{AC}^2}{2} = \overline{HB}\left(\overline{AG}\right)$$

And so again, see that when the angle between the legs of the machine is greater than 90 degrees the inverse proportion holds exactly the same way with the proportionality constant being half of the difference of the squares of the legs. Next it will be shown that the three last arrangements of the legs hold to exactly the same proportionality constant.

90 degrees

Here is the case for the angle between the legs at exactly 90 degrees. When the segments of the Inverse Proportion Machine are oriented this way the point H coincides with the point A since point H is the point where the perpendicular to the base of the Inverse Proportion Machine, $\overline{AB}$, reaches up to touch point C.



By the Smaller Hypotenuse Property of Right Triangles:

$$\overline{CB}^2 = 2\overline{AB}\left(\overline{GB}\right)$$

By inspection:

$\overline{AG} = \overline{AB} - \overline{GB}$ and by rearranging:

$$\overline{GB} = \overline{AB} - \overline{AG}$$

Substituting for $\overline{GB}$:

$$\overline{CB}^2 = 2\overline{AB}\left(\overline{AB} - \overline{AG}\right) = 2\overline{AB}^2 - 2\overline{AB}\left(\overline{AG}\right)$$

By inspection of the right triangle ΔABC and Pythagorus:

$$\overline{CB}^2 = \overline{AB}^2 + \overline{AC}^2$$

Substituting for $\overline{CB}^2$:

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AB}^2 - 2\overline{AB}\left(\overline{AG}\right)$$

Rearranging:

$2\overline{AB}\left(\overline{AG}\right) = 2\overline{AB}^2 - \overline{AB}^2 - \overline{AC}^2$ and by arithmetic:

$$2\overline{AB}\left(\overline{AG}\right) = \overline{AB}^2 - \overline{AC}^2$$

$$\overline{AB}\left(\overline{AG}\right) = \left(\overline{AB}^2 - \overline{AC}^2\right)/2$$

Since point A and point H coincide for this arrangement of the Inverse Proportion Machine we can define the point as being either A or H as we wish. And so the equation may be stated as:

$$\left(\overline{HB}\right)\left(\overline{AG}\right)=\frac{\overline{AB}^2-\overline{AC}^2}{2}$$

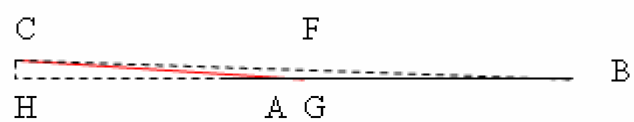
Again, the product of the two segments are equal to the same constant as above.

<u>180 degrees</u>

Now for the case where the angle between the legs is 180 degrees:

Note that as the leg spins so that the far end of the leg $\overline{AC}$ is just about at 180 degrees but not quite, one can see what will happen when the angle reaches exactly 180 degrees. The perpendicular bisector of the connector will bisect the distance from C to B so that the point F will coincide with the point *G* . The point *C* will coincide with the point *H* as the leg straightens completely. The point *G* will be at the exact midpoint of the entire length $\overline{HB}$.

The resulting line will have the points that we are interested in, *H,A,G,andB.*



Note that due to the bisector $\overline{HG} = \overline{GB}$.

When the Inverse Proportion Machine segments reach 180 degrees the points coincide as described above so that the segment of the machine, $\overline{AC}$ can be validly be defined as $\overline{AH}$. For the arrangements of the Inverse Proportion Machine we have seen so far, the product of the segments $\overline{HB}$ and

$\overline{AG}$ were shown to be equal to the half the difference of the squares of the segments $\overline{AB}$ and $\overline{AC}$:

$$\left(\overline{HB}\right)\left(\overline{AG}\right)=\frac{\overline{AB}^2-\overline{AC}^2}{2}$$

This is equivalent to :

$$\left(\overline{HB}\right)\left(\overline{AG}\right)=\frac{\left(\overline{AB}+\overline{AC}\right)\left(\overline{AB}-\overline{AC}\right)}{2}$$

Since at 180 degrees, segment $\overline{AC}$ is equal to segment $\overline{AH}$ the equation above becomes:

$$\left(\overline{HB}\right)\left(\overline{AG}\right)=\frac{\left(\overline{AB}+\overline{AH}\right)\left(\overline{AB}-\overline{AH}\right)}{2}$$

Now, if it can be shown that the above equation holds true at 180 degrees, the segments $\overline{HB}$ and $\overline{AG}$ will be proven again to behave according to the same proportion.

Examine the numerator $\left(\overline{AB}+\overline{AH}\right)\left(\overline{AB}-\overline{AH}\right)$. For convenience, let this quantity equal *x* and then solve for *x* to see if it equals $2\left(\overline{HB}\right)\left(\overline{AG}\right)$ as it did for the other cases of the Inverse Proportion Machine.

Let

$$\left(\overline{AB}+\overline{AH}\right)\left(\overline{AB}-\overline{AH}\right)=X$$

By inspection:

$$\left(\overline{AB}+\overline{AH}\right)=\overline{HB}$$

Also by inspection:

$$\overline{AG}+\overline{GB}=\overline{AB}$$

Also by inspection:

$$\overline{HG}-\overline{AH}=\overline{AG}$$

By bisector:

$\overline{HG} = \overline{GB}$

So substituting for $\overline{HG}$:

$\overline{GB} - \overline{AH} = \overline{AG}$

Substituting $\left(\overline{AG} + \overline{GB}\right)$ for $\overline{AB}$ and $\overline{HB}$ for $\left(\overline{AB} + \overline{AH}\right)$ in the first equation for *x* gives:

$\overline{HB}\left(\overline{AG} + \overline{GB} - \overline{AH}\right) = X$

by substituting $\overline{AG}$ for

$\overline{GB} - \overline{AH}$

the result is:

$\overline{HB}\left(\overline{AG} + \overline{AG}\right) = X$

so $X = 2\overline{HB}\left(\overline{AG}\right)$

Plugging back in for X which we defined to be $(\overline{AB}+\overline{AH})(\overline{AB}-\overline{AH})$:

$2(\overline{HB})(\overline{AG}) = (\overline{AB}+\overline{AH})(\overline{AB}-\overline{AH})$ so that :

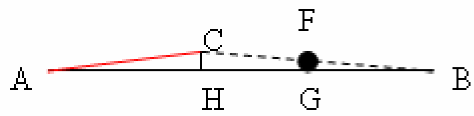
$(\overline{HB})(\overline{AG}) = \frac{(\overline{AB}+\overline{AH})(\overline{AB}-\overline{AH})}{2}$ and since segment $\overline{AH}$ is equivalent to segment $\overline{AC}$ at 180 degrees, it is proven again that:

$(\overline{HB})(\overline{AG}) = \frac{(\overline{AB}+\overline{AC})(\overline{AB}-\overline{AC})}{2}$ when the angle between segments $\overline{AB}$ and $\overline{AC}$ is 180 degrees.

<u>Overlapping segments</u>

The last arrangement will be for when the legs overlap. The angle between the legs is 0 or 360 degrees:

As the legs come together, point *F* coincides with point *G* and point *C* coincides with point *H* :



Since point *F* and *G* are the bisector of $\overline{HB}$:

$$\overline{HG} = \overline{GB}$$

As before evaluate the expression

$$\left(\overline{AB} + \overline{AH}\right)\left(\overline{AB} - \overline{AH}\right)$$

as was done for the 180 degree case above:

Note that as the angle between the segments $\overline{AB}$ and $\overline{AC}$ becomes zero degrees:

$\overline{AH} = \overline{AC}$

$\left(\overline{AB} + \overline{AH}\right)\left(\overline{AB} - \overline{AH}\right) = X$

Let:

By inspection:

$\overline{AB} + \overline{AH} = \overline{HB}$

Also by inspection:

$\overline{AG} + \overline{GB} = \overline{AB}$

So substituting $\overline{HB}$ for $\overline{AB} + \overline{AH}$ and $\overline{AG} + \overline{GB}$ for $\overline{AB}$:

$X = \overline{HB}\left(\overline{AG} + \overline{GB} - \overline{AH}\right)$

Examine the term $\overline{GB} - \overline{AH}$ in the parentheses above:

Since $\overline{HG} = \overline{GB}$:

$\overline{GB} - \overline{AH} = \overline{HG} - \overline{AH} = \overline{AG}$ by inspection of the figure above.

So substituting $\overline{AG}$ for $(\overline{GB} - \overline{AH})$: $X = \overline{HB}(\overline{AG} + \overline{AG})$

So: $X = 2\overline{HB}(\overline{AG})$

And now plug back in for X which we defined to be $(\overline{AB} + \overline{AH})(\overline{AB} - \overline{AH})$:

$$(\overline{AB} + \overline{AH})(\overline{AB} - \overline{AH}) = 2\overline{HB}(\overline{AG})$$

and so

$$\overline{HB}(\overline{AG}) = \frac{(\overline{AB} + \overline{AH})(\overline{AB} - \overline{AH})}{2}$$

And since for the angle of zero degrees between segments, $\overline{AH} = \overline{AC}$:

$$\overline{HB}(\overline{AG}) = \frac{(\overline{AB} + \overline{AC})(\overline{AB} - \overline{AC})}{2}$$

This is the same equation that we found for the product of the segments $\overline{HB}$ and $\overline{AG}$ for all other arrangements of the segments $\overline{AB}$ and $\overline{AC}$ of the Inverse Proportion Machine.

A hint of what is to come:

It is evident that for any angle between the segments $\overline{AB}$ and $\overline{AC}$ the segment $\overline{HB}$ is inversely proportional to the segment $\overline{AG}$.

We have a machine that can create segments of a line that are inversely proportional to each other. This is of tremendous importance to our purpose. We can assign the inversely proportional segments to represent properties of planetary movements that are known to be inversely proportional to each other.

By the requirement that equal areas are swept in equal times we have our two inversely proportional properties. One is the radius, r or distance to the Sun. The other is the tangential velocity, $V_{\tan gential}$.

The Inverse Proportion Machine will allot these properties their proper magnitudes, initiating the *a priori* proof that planets travel elliptical paths.

To state this again, much more boldly; The Inverse Proportion Machine might command precisely the movement of a planet in its orbit. All we have to find is one property of motion that must always be inversely proportional to another property. If the properties can be represented by segments on the Inverse Proportion Machine we will be on the way to seeing that invisible machine which governs the movement of the planets. We will call the machine a "hododyne". And we will see things in a new way.