By **Convention** : The Circle's Hodograph

In this chapter our aim is to do something similar to drawing east and north on the same straight line. Sure that is impossible in the literal sense. But it is rewarding. How do we do such a thing? Presto; We set up a convention - a process that must be applied consistently.

In this chapter the hodograph will be introduced. It was first described by Sir William Rowan Hamilton. The hodograph is a diagram whose line segments represent motion and position.

Let's pause for a few words about Sir William Rowan Hamilton. He was born in 1805 in Ireland.   His many contributions to mathematics include a framework of multidimensional analysis that is considered to be a precursor to Einstein's Theory of Relativity.   Our connection to him is through the 1846 publication of his mathematical invention, the hodograph.   He presented the hodograph as a method that could be used to geometrically correlate Isaac Newton's Inverse Square Law of Force with the elliptical shape of orbits.

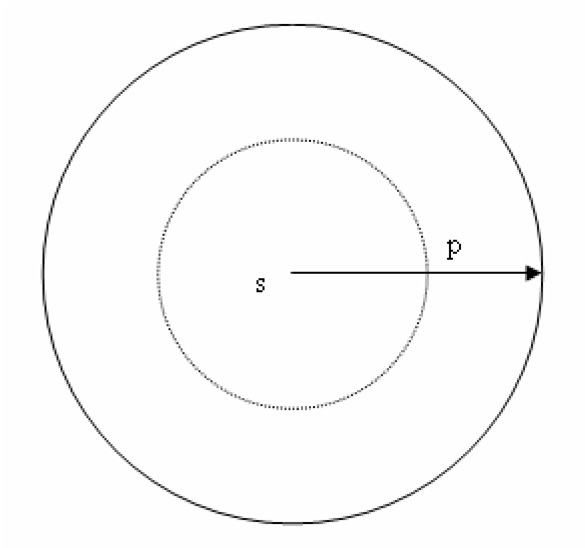
Now we will turn our attention to the concept that a convention can be devised to serve our purposes. The convention will help us to build a hodograph. For simplicity, in this chapter the analysis of hodographs will be limited to those which represent circular orbits.

An introduction to the hodograph

Hodographs are diagrams that display two things at once. They display the position of a planet and the velocity of the planet at any given moment. Velocity is displayed in the strict sense that we must be able to tell not only the speed of the planet but also its direction of travel.

The hodograph diagram for a circular orbit is instructive. Let us apply what we learned in the previous chapter. We showed using geometry and logic that an orbiting planet sweeps out equal areas in equal times. Thus a planet in a circular orbit must travel at constant speed along the circumference of the circle. Only in this manner - that is to say at constant speed - could the planet sweep out equal wedge shaped areas in equal small time periods.

Let's make our diagram. As the planet rotates around the Sun, let the position of the planet be marked on a line to the Sun - and let that line containing the planet rotate with the planet as the planet orbits. Let that line be called the position line. Let the same line also have a designated portion of specified length, indicated by an arrow, that represents the tangential velocity of the planet. Tangential velocity is the velocity at a right angle to the line toward the Sun. But wait,- we want our single line to contain information about things that are at right angles to each other - the position line and the tangential velocity arrow of the planet. How can we do that? There is an obvious paradox which is that the line connecting the planet to the Sun can not literally contain the arrow that represents tangential velocity because tangential velocity is actually perpendicular to the line. To resolve the paradox we make a note that the paradox exists, and then we decide how to deal with it. We will decide on a convention that will be used to interpret the arrangement of tangential velocity and planetary position when they are indicated on the same straight line. Our convention will be that we will always know that in order to find the true direction of velocity, we must rotate the velocity arrow's direction 90 degrees from where it is indicated on the hodograph diagram. That is simple enough.



The convention of 90 degree rotation

In the figure above, the planet travels a circular orbit represented by the inner dashed circle. The planet is at position p on the dashed circle and the distance and direction relative to the Sun are represented by the position and direction and length of the segment $\overline{sp}$. The tangential velocity when the planet is at position p is designated by the arrow which reaches the outer circle. As the planet rotates around the Sun, as it travels along the dashed circle, the magnitude of tangential velocity does not change since the distance to the Sun is constant for a circular orbit and the planet must sweep equal areas in equal times. So as the direction of tangential velocity changes, the tip of the arrow representing tangential velocity, whose magnitude does not change, traces out the solid large circle in the figure above. In other words an arrow of constant length representing a velocity of constant magnitude will sweep a circle as it rotates. So we have two circles on the diagram - one traced by the planet as it orbits - the dashed circle - and the solid circle traced by the tips of the tangential velocity arrows. We did not draw all the velocity arrows on the diagram - we drew only one for the planet at position p . But if we did , it is obvious that all the tips of the equal- magnitude tangential velocity arrows would lie on the solid circle. By our convention, the direction of each tangential velocity arrow is 90 degrees away from the direction that it truly represents. In summary then, each velocity arrow is correct in magnitude but off by 90 degrees in direction.

Why should we do it this way? Why insist on representing tangential velocity and position on the same straight line? We just explained that they can not possibly be on the same straight line. So why should we want to take an apparently awkward view of the planet's properties - a view from a perspective that can not exist? The answer is that this arrangement will reward us.

First of all let us agree that we are not breaking any rules if we define that the velocity direction must be rotated 90 degrees from where it is represented on the hodograph. We have every mathematical right to decide what our convention is going to be, as long as we stick to it. In other words we are stating that we are building a hodograph diagram and our convention is to draw the velocity always at 90 degrees from where it truly is.

So how will this arrangement reward us? It will reward us because there is a machine that can measure a planet's tangential velocity and the distance to the Sun. How does this machine dole out the information? The machine doles out the planet's position and the tangential velocity on the same straight line exactly as described in our convention above. Very fitting indeed.

What is this machine? It is a novel invention, the Inverse Proportion Machine. It will be described in the next two chapters.