Equal Areas Swept in Equal Times

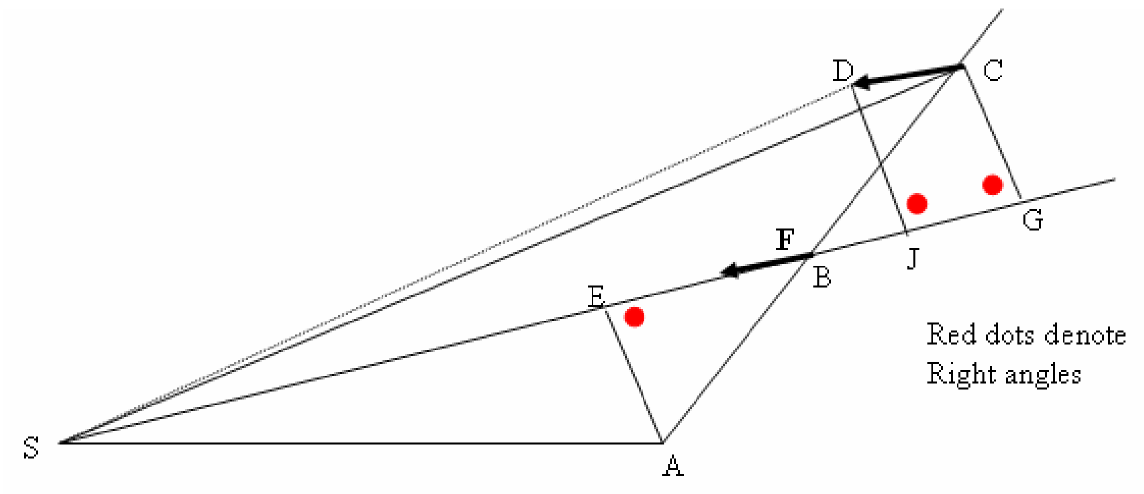
Let's pause to differentiate between two types of knowledge. Johannes Kepler was interested in discovering the causes behind the movements of planets. He was in search of *a priori* explanations - that is to say that he wanted to understand orbits in a theoretical way. His success was however more empirical in nature than *a priori* in nature; In other words, his discoveries were based on the experience of observing the positions of the planets in the sky and fitting these observations into a theory. When we start with observations and build a theory we are using empirical methods. On the contrary if we can build a theory from scratch without using observations our methods are *a priori* and the nature of that knowledge gained is more purely conceptual. The intended point is not that either type of knowledge is better. Kepler's Laws concerning planetary motion are perfect in practice and were exquisitely derived despite being empirical in nature. But perhaps an *a priori* understanding is deeper since it relates to the unseen driving factors behind the observations that we make. *Orbits Explained* aims to present orbits using *a priori* knowledge. It is my sense that there has been a gap in the literature and flow of knowledge. Kepler's empirical Planetary Laws are used as the starting point for scientific proofs. But the starting point could be more fundamental if it is the *a priori* demonstration that orbits are ellipses and that gravitational force is inversely proportional to the square of the distance. And so that is the gap where *Orbits Explained* aims to fill a void.

Kepler's Second Law states that planets sweep out equal areas in equal times as measured from the Sun. It might not ever have become evident to him since so many conditions had to be favorable for the discovery. First it was necessary to have data that was collected by Tycho Brahe concerning the positions of the planet Mars. Brahe invited Kepler to work with him as an interpreter of the extensive astronomical data that he collected. Although Kepler was grateful for the monumental opportunity to analyze Brahe's findings he was disappointed that Brahe was only willing to share some of the data. The data was even more problematic after Brahe's death due to legal battles with the Brahe family regarding possession and access. Imagine the time and anxiety involved in this process.

Once the data was obtained the difficult analysis ensued. Kepler was able to chart a course for Mars but noticed an eight minute discrepancy in his predictions. Desiring exactness he sought ways to improve on his methods. It became evident to him that Brahe's observations were from the Earth's position and that the actual path and shape of the Earth's orbit was unknown and was thus a likely source of error. A brilliant insight led him to calculate the orbit of the Earth using the preliminary orbit of Mars that he had calculated from Brahe's data as a reference orbit. When the orbit of the Earth was calculated Kepler noticed that by comparing perihelion to aphelion, he could see that the distance to the Sun was inversely proportional to the Earth's velocity. He knew, as we will see in our own way in Chapter 8, via geometrical demonstrations, that this implied that equal areas were swept in equal times. Kepler was able to extend this property to other positions along the orbit thus securing his Second Law: Planets sweep equal areas in equal times.

A small irony with regard to Kepler's Second Law is that Isaac Newton was later able to demonstrate it in *a* *priori* fashion, without the need for astronomical observations. Rather than trying to decide which demonstration is more brilliant it probably suffices to consider them both to be extraordinary in their own way.

Newton's method is now presented:



One diagram can reveal in a priori fashion that a planet sweeps out equal areas in equal times as it orbits the Sun. This one diagram is a composite of a number of diagrams that are found in modern mathematical texts and in Isaac Newton's writings. In this single diagram it will be shown first that a moving object will sweep out equal areas in equal times as it moves past a fixed point in space. Then it will be shown within the same diagram that if there is a force attracting the moving body towards the stationary object, the moving body will still sweep out equal areas in equal times relative to the stationary point.

Equal areas in the absence of force

First take the case of the moving object unaffected by a point in space as it moves past. It is moving at constant velocity since it is not acted on by any force. In equal time periods it therefore moves equal distances along a straight line. The moving body starts at point a. In equal times it will travel a distance from a to b that is equal to the distance from b to c. Note two right triangles, aeb and bgc. Note two equal angles eba and cbg. Since the sum of angles in every triangle is 180 degrees, the third angle of each of the two triangles aeb and bgc must be equal. Since these two triangles also share a hypotenuse the two triangles must be identical. Thus their heights ea and cg must be equal. Now notice that our moving object sweeps triangle sab and then triangle sbc in equal times as it goes from a to b and then from b to c. The area of a triangle is one half the base times the height. The base of the triangles of area swept is the same since it is the shared segment sb. The height of each of the triangles are the same since we showed $ea = cg$. Therefore the triangles sab and sbc are equal and are swept in equal times.

Thus the moving body indeed sweeps equal areas in equal times as it moves past a stationary point.

Equal areas in the presence of force

Now on to the situation of a force directed toward the stationary point. Let the stationary point be the Sun and the moving object, a planet. Let the diagram above show what happens as the planet moves from point b in a time period that we set as equal to the time it took to travel from a to b and then from b to c in our previous situation of no gravitational force. It must be emphasized that all the time periods are equal. If unaffected it would go to point c but instead it is acted upon in the instant it is at b by a force directed toward the Sun at s . This imparts a distance moved toward the Sun. It is important to consider the spatial situation to be one that involves great distances so that the force does not ruin the flight path of the planet by causing a crash landing into the Sun. Now instead of drawing this distance first toward the Sun from point b and then imposing the distance the planet would travel if unaffected, draw the distance traveled in the other order. This is justifiable. If a man walks one block east and one block north he will wind up in the same place regardless of which block he walks first. Thus let the planet travel from b to c which is what would happen if the planet were unaffected by the gravity of the Sun and then from c to d which is the direct result of the force of the gravitational pull of the Sun on the planet. Keep directions consistent by considering the following relationships. Notice at b the force is along the line sb. So after the planet reaches the point c its distance traveled due to the gravitational force must be parallel to the line sb or else the requirement that direction be exactly preserved would be neglected. By inspection, the lines that contain the segments cd and sg must be parallel since we require that cd is parallel to sg. Thus the heights cg and dj must be equal.

Therefore the triangles sbc and sbd share a base sb and their heights , cg and dj are equal.

So the area of triangle sbc is equal to the area of triangle sbd. We already showed that triangles

sbc and sab have equal areas. So all three triangles have equal areas sab, sbc, and sbd.

Thus the planet acted on by the gravitational force of the Sun sweeps out equal areas in equal times.