Changing Orbits

We are almost at the final chapter. As mentioned in Chapter 39, instead of concluding at Chapter 33 at which time *a priori* proofs of Kepler's Laws were presented, this book continued in order to give an *a priori* proof of The Energy Equation. This allows us to go beyond merely understanding why orbits are ellipses and why they behave according to Kepler's Laws. The Energy Equation brings us the additional dimension of understanding which relates to how orbits function - that is to say - how they might change if one aspect of the orbit is suddenly altered. . Furthermore, if only a few aspects of an orbit are known, the Energy Equation might be able to determine for us other aspects of the orbit. By inspecting the terms of the Energy Equation, we can indeed see that it might help us to understand how a change in one term might affect a change in another term:

$$V_{esc}^{2} - V_{3}^{2} = \frac{GM}{a}$$

Recall that $V_{esc}^{2} = 2V_{c}^{2}$ where V_{c} is the velocity of a hypothetical planet in a circular orbit at a given distance from the Sun and that V_3 is the total velocity of the actual planet.

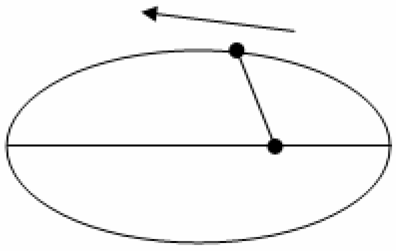
By inspection of the Energy Equation above, noting that for a solar system GM does not change, we might be able to predict how the length of the semimajor axis will change if the total velocity is suddenly altered. Keep in mind that at the moment the total velocity is altered the distance to the Sun has not yet changed. Since the distance to the Sun has not yet changed, the velocity of a hypothetical planet in a circular orbit at that distance has not changed either. Thus the escape velocity which is dependent on the hypothetical circular orbit's velocity has not changed either. In other words, the first term of the Energy Equation has not changed. That leaves only two variables. The first is the total velocity and the second is the length of the semimajor axis. Therefore we can predict the new semimajor axis length that results from a sudden change in total velocity. Such a scenario could occur in the setting of a perturbation of a planet by the gravitational pull of another planet. Alternatively, if the orbiting body is a space ship instead of a planet, the total velocity could change due to a sudden thrust commanded by the astronauts.

Furthermore, Kepler's Third Law allows us to predict the new period of the resultant orbit once the new semimajor axis length is found.

For illustration, a brief sampling of the application of the Energy Equation follows. An example of thrust applied to a space ship, an example of the determination of a comet's orbit, and an example of a planet perturbation will be presented.

A Space Ship Changes Orbit

For the first scenario, a space ship orbiting around the Sun will be examined. Assume that the initial orbital velocity, period, and position of the ship are known. That allows calculation of GM which, as will become evident, is necessary for predicting new orbits.



Suppose the space ship is in an orbit as represented above with a known eccentricity of 0.6. The known semimajor axis is three astronomical units, 3 a.u.. (An astronomical unit is defined as the average distance between the Earth and the Sun.) The ship is known to be 115 degrees from perihelion. Its period can be calculated from Kepler's Third Law. The period is approximately 5.2 years since $T \propto R^{\frac{3}{2}} \propto 3^{\frac{3}{2}}$. Now, $3^{\frac{3}{2}} \approx 5.2$, so the period is 5.2 times longer than the Earth's period. The semilatus rectum is $p = a\left(1 - e^2\right) = 3\left(1 - 0.6^2\right) = 1.92$ astronomical units. The radius to the Sun in the position above is given by $r = \dfrac{p}{1 + e\cos\theta} = \dfrac{1.92}{1 + (0.6)(\cos 115)} = 2.57$ astronomical units. The areal velocity is h, twice the area swept in unit time. The length of the semiminor axis is derived from the semilatus rectum and the semimajor axis lengths. $b = \sqrt{ap} = \sqrt{3 \times 1.92} = 2.4$

astronomical units. Twice the area swept in one period of the orbit is $h = 2\pi ab \div 5.2 = 8.7 \quad a.u.^2$ per Earth Year.

From $h = \sqrt{GMp}$, derive $GM = \frac{h^2}{p} = \frac{8.7^2}{1.92} = 39.4$. The velocity in the position above is given by

$V_{total}{}^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) = 39.4\left(\frac{2}{2.57} - \frac{1}{3}\right) = 17.5$ so V_{total} for the position above is equal to $\sqrt{17.5} = 4.18$ a.u. per year.

Now suppose that the space ship's engines fire to deliver a thrust that changes the velocity of the ship so that the velocity is increased to 5 a.u. per year without changing the direction of the velocity.

The Energy Equation will reveal the new semimajor axis length. Then logic and the formula above, relating θ to radius, will give the orientation of the new semimajor axis relative to the old one.

The circular orbit at a distance of 2.57 astronomical units from the Sun has a velocity given by

$V = \sqrt{\frac{GM}{r}} = \sqrt{\frac{39.4}{2.57}} = 3.92$. The Energy Equation reveals that

$2(3.92)^2 - 5^2 = \frac{39.4}{a}$ for the new orbit. Then, solving for the new semimajor axis, a, the result is 6.88 astronomical units. The new orbital period will be given by $6.88^{\frac{3}{2}} = 18.0$ years.

The velocity vector will not change direction when thrust is applied according to the premise of our scenario. The proportion between total and tangential velocity will be the same as it was before the thrust since that proportion is dependent only on the unaltered direction to the Sun.

Compute the ratio of tangential to total velocity before the thrust. It was calculated that h is equal to 8.7 and that r is equal to 2.57 astronomical units.

$V_{\tan gential} = \frac{h}{r} = 3.38$ a.u. per Earth Year. So $\frac{V_{\tan gential}}{V_{total}} = \frac{3.38}{4.18} = .81$

For the new orbit it is given that the new total velocity is 5.0 a.u. per year. So applying the ratio just obtained:

$V_{\tan gential} = .8055(5.0) = 4.05$ after the thrust.

r is unchanged at the moment of thrust at 2.57 a.u..

After thrust $h = r \times V_{\tan gential} = 2.57(4.05) = 10.4$

It is known that $h=\sqrt{GMp}$ so that $p=\frac{h^2}{GM}=\frac{10.4^2}{39.4}=2.75$ a.u..

It was calculated that the new a is 6.88 a.u..

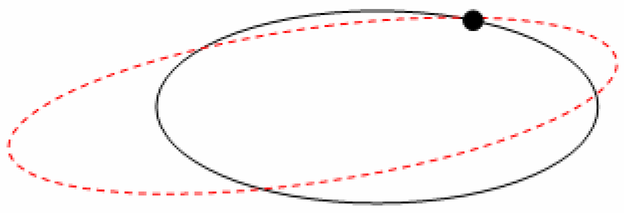
It is known that $p=a\left(1-e^2\right)$ so solving for eccentricity we get $e=\sqrt{1-\frac{p}{a}}=\sqrt{1-\frac{2.75}{6.88}}=.77$

Solve for $\cos\theta$ in the polar equation above:

$$\cos\theta=\frac{p-r}{er}=\frac{2.75-2.57}{.77(2.57)}=.09$$

From the trigonometric tables of angles and their cosines $\theta=85$ degrees.

So before the moment of thrust the ship was 115 degrees from perihelion. Then at the moment of thrust the ship finds its velocity increased and 85 degrees from perihelion. It is suddenly in a new orbit of semimajor axis length 6.88 a.u. compared to the original 3 a.u.. The new orbit is rotated 30 degrees relative to the initial orbit. The new orbit was successfully predicted after a thrust was applied.



In general terms the initial orbit in black and the new orbit in red are approximated for the thrust scenario described above.

This is a good time for a philosophical pause in order to ask the question, "Why is it valid to assume that if a planet's orbit is suddenly altered by a thrust, the planet will still return to the same position in the new orbit?" This is in fact an interesting question. The reasoning behind the return to the same position is that for any orbit a planet returns to a given position ad infinitum as directed by the hododyne. In fact each position along the orbit has its own unique velocity that is reproduced each time around. Applying a thrust either simply puts the planet in a new orbit at a certain position and velocity or causes the planet to escape. If it stays in orbit, then as with any orbit due to its new hododyne, the thrusted planet will return to the location of thrust ad infinitum. A further consideration is that the Energy Equation tells us if the planet will escape from the Sun instead of returning to the same spot; if the thrust increases velocity enough, the solution to the Energy Equation will be a negative number. The Energy Equation tells us how much energy is lacking for escape. The Energy Equation is the mathematical expression of the Planetary Capture Law. A negative number tells us that the amount of deficit of energy is negative; this is the same as stating that there is a surplus of energy – too much energy and so the planet can not stay in orbit. It is no longer captured.

What can be said about the Energy Equation and the situation of free fall directly towards the Sun? Certainly this will be a collision and not an orbit. There is an obvious commandment concerning orbits. It is not acceptable to fall directly towards the Sun. In philosophical physical jargon, “Do not let your tangential velocity be zero.”

Predicting a Comet's Orbit

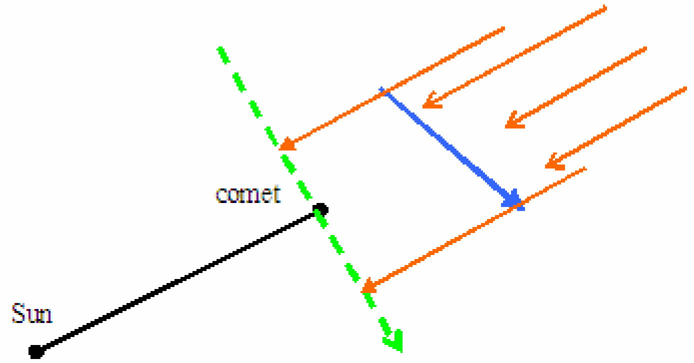
In the second scenario there is a hypothetical comet that is discovered on its approach to the Sun. Its position and velocity relative to the Sun are calculated based on multiple astronomical observations. It is necessary to know the value of GM for the Sun in order to predict the orbit of the comet. It is convenient to use the value that was obtained in the scenario above concerning the space ship.

Suppose the distance of the comet is observed to be 250 astronomical units. Suppose its velocity is calculated to be 0.120 astronomical units per year. Apply the Energy Equation to determine the length of the semimajor axis of the orbit:

The circular orbit at a distance of 250 a.u. has a velocity of $V_c = \sqrt{\frac{GM}{r}} = \sqrt{\frac{39.4}{250}} = .4$ a.u. per year. The Energy Equation states:

$$2V_c^2 - V^2_{total} = \frac{GM}{a} = .32 - .0144 = .306$$

Solving for semimajor axis, $a = 129$ astronomical units. The period of the comet's orbit is $129^{\frac{3}{2}} = 1465$ Earth Years.



In the diagram above let the total velocity of the comet be represented by the blue arrow. The direction of tangential velocity is demonstrated by the dashed green arrow. The red imaginary rays, parallel to the imaginary line joining the comet to the Sun, are used in the shadow method to determine the magnitude of the tangential velocity. The magnitude of tangential velocity is represented by the length of the green dashed arrow between the two tips of the red arrows falling on it. Suppose the observations revealed that, as in the diagram above, the comet's total velocity made an angle of 20 degrees with the line perpendicular to the line to the Sun. In other words the total velocity made an angle of 20 degrees with the tangential velocity component. The tangential velocity by the shadow method and by the cosine definition is therefore given by:

$$\cos 20 = \frac{V_{\tan gential}}{V_{total}} = \frac{V_{\tan gential}}{.120}$$

Solving for tangential velocity since the cosine of 20 degrees is equal to 0.94 we get:

$V_{\tan gential} = .1128$ a.u. per year.

Calculate areal velocity:

$h = r \times V_{\tan gential} = 250(.1128) = 28.2$ square a.u. per year

Obtain the length of the semilatus rectum:

$$p = \frac{h^2}{GM} = \frac{795}{39.4} = 20.2 \text{ a.u.}$$

Calculate eccentricity:

$$e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{20.2}{129}} = .918$$

Obtain the angle relative to the line connecting the Sun and perihelion position:

$$\cos\theta = \frac{p - r}{er} = \frac{20.2 - 250}{.918(250)} \approx 1.0 \text{ so that } \theta \approx 180 \text{ degrees.}$$

The comet was thus discovered when it was farthest from the Sun in its orbit at its aphelion position. It will be swinging by the Sun in about 732 years (half its orbital period).

Mars Perturbs the Earth

This final scenario examines the perturbation of the Earth due to Mars over a one day span of time. Known approximate measurements of the orbits of the two planets and an approximation of the mass of Mars will be utilized. The Earth's semimajor axis is equal one astronomical unit and the eccentricity is approximated to be zero for simplicity. Under the condition of zero eccentricity for Earth it is easy to avoid having to deal with the awkward diagrams that would be necessary in order to show the exact direction of total and tangential velocity of the Earth for the position where perturbation is examined.

Examine a day for which Mars is at its perihelion position and also in line with the Earth and the Sun so that it is at a relatively close approach to the Earth. The eccentricity of Mars' orbit can be approximated to be .09 and the semimajor axis to be 1.5 astronomical units. The mass of Mars can be taken to be 6.4×10^{23} kg.. The mass of the Sun can be taken to be 2×10^{30} kg.. The mass of the Earth can be taken to be 6×10^{24} kg.. The Gravitational Constant can be taken to be 6.7×10^{-11} distance in meters and mass measured in kilograms. The semimajor axis for the orbit of Mars can be taken to be 2.3×10^{11} meters. The perihelion distance of Mars is given by :

$$r_{perihelion} = a(1-e) = 2.3 \times 10^{11}(1-.09) = 2.09 \times 10^{11} \text{ meters.}$$

The distance of the Earth to the Sun can be taken to be 1.5×10^{11} meters.

The distance between the planets when they are lined up as above is

$2.09\times10^{11}-1.5\times10^{11}=5.9\times10^{10}$ meters

The force exerted by Mars on the Earth is

$$F=\frac{GMm}{r^2}=\frac{6.7\times10^{-11}\left(6.4\times10^{23}\right)\left(6.0\times10^{24}\right)}{\left(5.9\times10^{10}\right)^2}=7.3\times10^{16}$$

It was shown that $F=ma$ so the acceleration of the Earth due to Mars is $\frac{GM}{r^2}$ where M represents the mass of Mars. The acceleration is thus:

$acc=\frac{6.7\times10^{-11}\left(6.4\times10^{23}\right)}{\left(5.9\times10^{10}\right)^2}=1.23\times10^{-8}$ kilometers per second per second.

Acceleration is defined to be $\frac{\Delta v}{\Delta t}$. The change in velocity if we assume that at the beginning of the day the velocity induced by Mars is zero is the final velocity achieved due to Mars. In a day there are 8.64×10^4 seconds.

So the final velocity toward Mars achieved is $1.23 \times 10^{-8}\left(8.64 \times 10^4\right) = 1.06 \times 10^{-3}$ kilometers per second.

The distance traveled by the Earth is $2\pi \times 1.5 \times 10^{11} = 9.4 \times 10^{11}$ kilometers. The time it takes to travel this distance is one year or 3.15×10^7 seconds. The velocity is essentially tangential for the approximately circular orbit and is $V = \frac{9.4 \times 10^{11}}{3.15 \times 10^7} = 2.9 \times 10^4$ kilometers per second.

So the initial velocity of 2.9×10^4 kilometers per second is subjected to a change in velocity at a right angle to initial velocity in the amount of 1.06×10^{-3} kilometers per second. The final velocity is given by:

$$V_f = \sqrt{\left(2.9 \times 10^4\right)^2 + \left(1.06 \times 10^{-3}\right)^2}$$

$$V_f^{\ 2} = 841000000.000001124 .$$

$$V^2{}_{initial} = 841000000$$

Now apply the Energy Equation to see what happens to the length of the semimajor axis of the Earth's orbit and the length of the Earth Year.

The new semimajor axis will be estimated after the perturbation, followed by an estimate of the new orbital period.

These calculations are estimates because of simplifications regarding the direction and magnitude of the force between the planets. It is assumed that the change in force is not large enough to consider since the distance between the planets is fairly stable. It is also assumed that the force acts in the same direction even though the Earth moves approximately one degree around in its orbit during the day that we are considering.

The Energy Equation will be applied twice to give the semimajor axis length. This is done for the orbit before perturbation even though a certain value was assumed for the semimajor axis at the onset of our scenario. In this way the Energy Equation more accurately shows what changes a perturbation brings about.

Before perturbation:

$$2V^2{}_{circle} - V^2{}_{total} = \frac{GM}{a}$$

$$2\left(2.9\times10^4\right)^2 - \left(2.9\times10^4\right) = \frac{\left(6.7\times10^{-11}\right)\left(2\times10^{30}\right)}{a}$$

Solving for semimajor axis before perturbation we get $a = 159334126040.42801$ meters

After perturbation:

$$2\left(2.9\times10^4\right)^2 - 841000000.000001124 = \frac{\left(6.7\times10^{-11}\right)\left(2\times10^{30}\right)}{a}$$

Solving for semimajor axis after perturbation we get $a = 159334126040.42827$ meters.

The ratio of semimajor axes is $\frac{159334126040.42801}{159334126040.42827}$. The ratio of the periods is the preceding fraction to the $\frac{3}{2}$ power.

Thus the perturbed orbit has a period longer than the initial orbit by the factor:

1.000000000000002448

Assuming 365 days in a year and 31536000 seconds in a year the number of seconds in the orbit after perturbation is:

31536000.00000007719 seconds.

The new perturbed orbit is roughly eight one hundred millionths of a second longer than the initial orbit.

Our Energy Equation reveals that an Earth Year is an extremely stable quantity. It is intuitive that as we go farther out in the solar system the velocities induced by planet perturbations are more significant in comparison to the slower orbital velocities. The changes in orbital periods and semimajor axes are therefore more significant as the distance from the Sun increases. But still, it is evident that the Earth's orbital period is remarkably stable. In other words there are plenty of years ahead in which to look for peace.

Postscript to the Energy Equation: Implications for an Ether

There has been a conjecture historically about the presence of a tangible material filling all space, the so called ether. It would follow from the simple empirical observation that orbits exist that such an ether could not be present. If there were an ether, it would create friction or drag so as to disturb the match between tangential velocity and radius to the Sun so that all planets would travel too slowly to stay in orbit indefinitely. Planets would spiral inward towards the Sun. The only way a planet could stay in orbit indefinitely would be to have an applied thrust to continually negate the effect of the ether’s drag. A practical consideration does arise regarding the energy of escape in the sense that while calculating the escape velocity for a rocket, the drag of the Earth’s atmosphere must be taken into consideration.