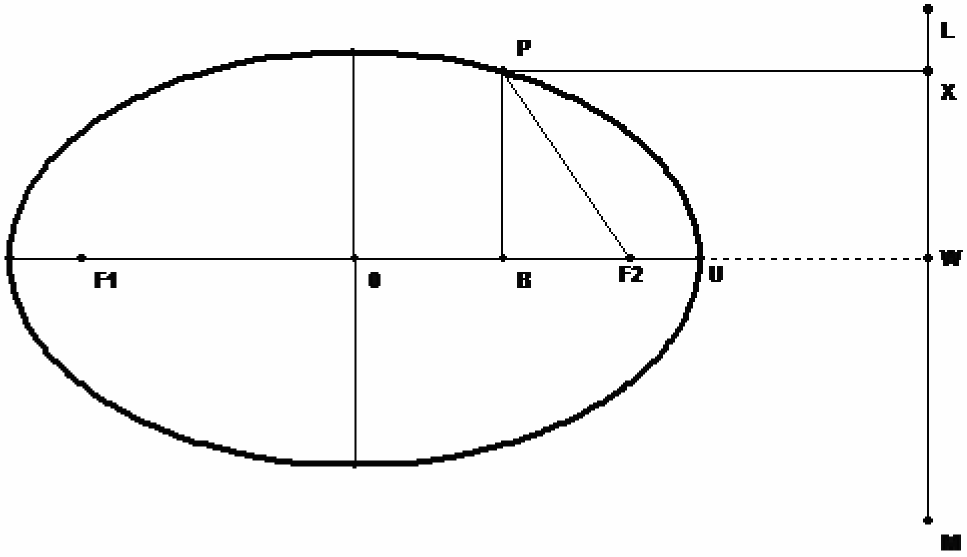
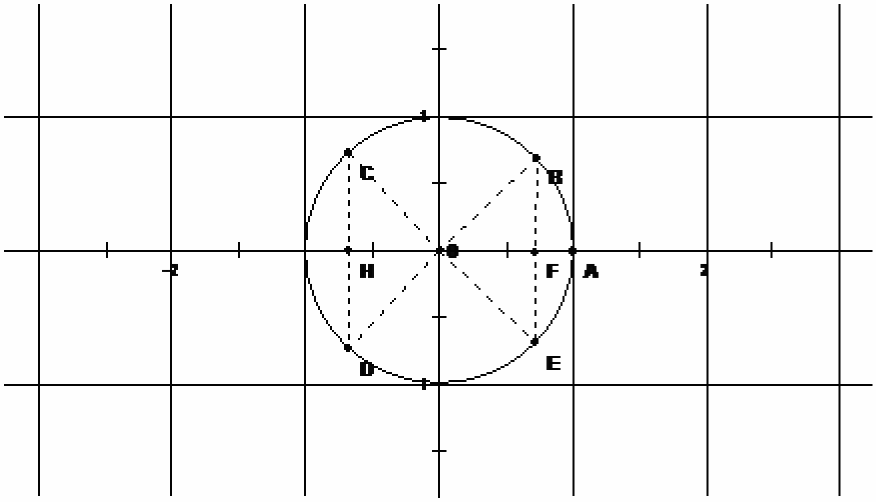
Polar Coordinates

This chapter demonstrates how to express the planet's position in polar coordinates. In the next chapter, polar coordinates will be helpful in one of the applications of the Energy equation. Credit in this chapter goes to mathematical texts. There is a way to express the position of a point in space by specifying its distance from a focus and its direction. We refer to the distance as the radius and the direction as the angle. The angle is measured as if viewed from the focus. For orbits the focus is the Sun. In the figure below, the distance from F2 to *P* is the radius to the planet from the Sun.   The angle is measured by convention from the line that would connect the planet to the Sun when the planet is at perihelion, that is, when the planet is at its closest distance to the Sun. In the figure this is the angle *WF2P* marked Ø.



Recall that for a right triangle, the cosine of an angle is equal to the length of the adjacent side of the triangle divided by the length of the hypotenuse of the triangle. Ø The cosine of an angle, Ø ,greater than 90 degrees is the same as negative cosine(180 – Ø). So the cosine of Ø in the figure above is equal to the negative of the cosine of the angle ***BF2P*** . We can show how to determine the cosine of angles greater than 90 degrees by using the unit circle and noting that the cosine of any angle is defined as the x value of the position on the unit circle corresponding to the angle.



We see that for the angle ***BOF*** that the cosine is ***OF/OB***. Since ***OB*** is equal to one in the unit circle, cosine of ***BOF=OF.*** Note that ***OF*** is equal to the x value of the position ***B*** on the unit circle of radius equal to one. Carrying the analogy to the position ***C*** on the circle the cosine of angle ***AOC*** is equal to the x value at ***C*** which is equal to the length of ***OH*** in the negative direction. Compare this to the cosine of angle ***COH.*** It is equal to ***OH/OC*** which equals ***OH*** since ***OC*** is equal to one. So we see the cosine of angles that add up to 180 degrees are negatives of each other.

Note that in the figure, the cosine of angle ***BF2P*** is equal to ***BF2/F2P.*** By the above demonstration , since angle Ø and angle ***BF2P*** sum to 180 degrees, the cosine of Ø is negative ***BF2/F2P.***

Since ***F2P*** is the radius, *r*, to the planet:

$\cos\theta = -\cos BF_2P = -\frac{F_2B}{r}$ so that $F_2B = -r\cos\theta$. Now let $WF_2 = s$.

Note that by inspection of the figure:

$$WB = PL = WF_2 + F_2B = s + F_2B = s - r\cos\theta$$

We showed in the previous chapter a property of the directrix which states that the distance from the position on an ellipse to the directirix multiplied by *e*, the eccentricity, equals the length of the leg of the ellipse. By inspection of the figure:

$PL \cdot e = r$ or by rearranging, $r = e \cdot PL$

So $r = e(s - r\cos\theta) = se - er\cos\vartheta$

Rearranging:

$$r + er\cos\theta = se$$

Simplifying:

$$r(1 + e\cos\theta) = se$$

Rearranging:

$$r = \frac{se}{1 + e\cos\theta}$$

Now note that the distance *s* is equal to the distance from the center of the ellipse to the directrix (which we showed to be $a \div e$) minus the distance from the center of the ellipse to the focus (which is defined as being *c)*. So:

$$s = \frac{a}{e} - c$$

Notice the numerator of the polar equation for *r*.

Substitute for *s* in the expression *se*:

$$se = \left(\frac{a}{e} - c\right)e$$

Multiply out:

$$se = a - ce$$

Note that since $\frac{c}{a} = e$, then $c = ea$. So we can substitute:

$$se = a - eae = a - ae^2 = a\left(1 - e^2\right)$$

So the numerator in the polar equation for *r* changes from *se* to $a\left(1 - e^2\right)$

Thus:

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

Now we will proceed to show that the numerator is equal to the length of the semilatus rectum, p. We will do this by manipulating expressions for the eccentricity, e.

Recall that $e=\frac{c}{a}$ and that $c^2=a^2-b^2$ so that $c=\sqrt{a^2-b}$ and thus $e=\frac{\sqrt{a^2-b^2}}{a}$

Now square:

$$e^2=\frac{a^2-b^2}{a^2}=\frac{a^2}{a^2}-\frac{b^2}{a^2}=1-\frac{b^2}{a^2}$$

Now substitute for e^2 in the numerator of the polar equation for r :

$$r=\frac{a\left(1-\left(1-\frac{b^2}{a^2}\right)\right)}{1+e\cos\theta}=\frac{a\left(\frac{b^2}{a^2}\right)}{1+e\cos\theta}=\frac{\left(\frac{b^2}{a}\right)}{1+e\cos\theta}$$

Since we know that $p=\frac{b^2}{a}$:

$$r = \frac{p}{1 + e\cos\theta}$$

And so we have an equation that describes the position of the planet along its ellipse. In the next chapter, we will be particularly interested in the angle θ . Our Energy Equation will use the formula above to derive the elliptical orbit and to locate the planet when its velocity and position relative to the Sun are known.