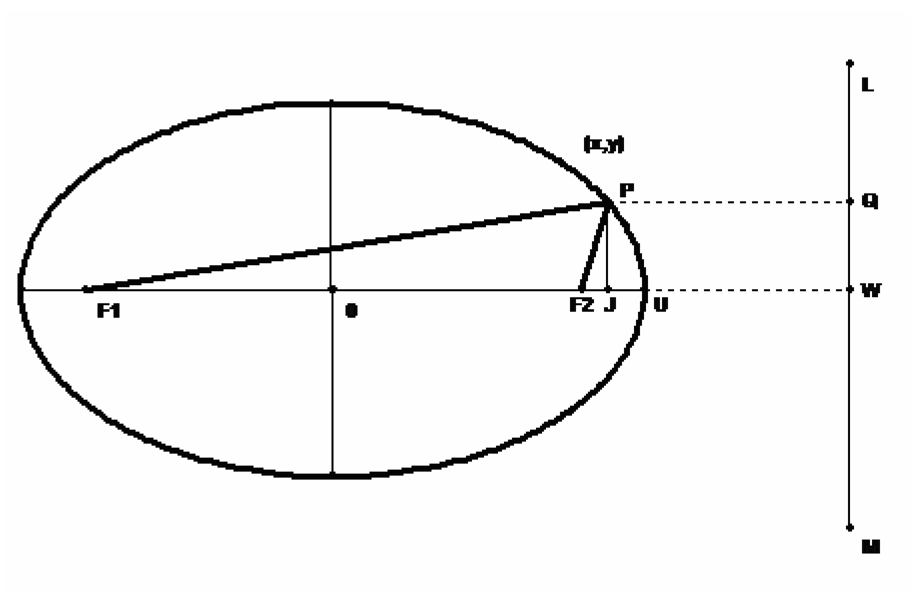
Directrix

In Chapter 42 concerning the Energy Equation we will need to be able to express the position of the planet in polar coordinates. Polar coordinates are used to express the position of the planet by measuring the angle which its radius makes with the major axis of the ellipse and by measuring the length of its radius to the Sun. As a preliminary we need to understand the concept of the directrix before demonstrating polar coordinates in Chapter 41. We give credit in this chapter to mathematical textbooks for the demonstration of the directrix.

Our previous definition of the ellipse concerns the string and tack properties as we showed in a Chapter 2. We can now demonstrate another way of defining an ellipse that involves a different property. We visualize an imaginary straight line which we will call the "directrix". The shortest distance from any position, P, along the ellipse to this imaginary line is important to us. We will use this distance to define an ellipse mathematically. This alternate definition of an ellipse states that the distance from a position, P, on the ellipse to a focus F_2 is a constant proportion of the distance of that position on the ellipse to an imaginary fixed line (the directrix). In figure 3 this states that as the point P moves along the ellipse, the length of the line from F_2 to P will be a fixed proportion of the length of the line from P to Q on the directrix. We will branch off from an equation we used in our first mathematical definition of the ellipse and proceed to show that the directrix exists and where to find it.



Place a vertical line at the x value of $\frac{a^2}{c}$. The distance from 0 to W is $\frac{a^2}{c}$. This will be shown to be a directrix of the ellipse. In other words we have drawn a straight line whose equation is $x = \frac{a^2}{c}$. Part of that straight line, *LM*, is shown

Figure 1

We showed in a Chapter 2 that:

$$a - \frac{c}{a}x = \sqrt{(x-c)^2 + y^2}$$

Observe the left side of the above equation and note that we express it as:

$$a-\frac{c}{a}x=\frac{ca}{ca}a-\frac{c}{a}x=\frac{ca^2}{ca}-\frac{c}{a}x=\frac{c}{a}\left(\frac{a^2}{c}-x\right)$$

So we can change the left side of the previous equation and restate it as:

$$\frac{c}{a}\left(\frac{a^2}{c}-x\right)=\sqrt{(x-c)^2+y^2}$$

This equation tells us that the directrix exists. It tells us where to find it. It also satisfies the directrix definition of an ellipse. Here is how to see these properties:

Note that in figure 1 that the distance from the position on the ellipse to the focus is the leg, PF_2, which we know is :

$$\sqrt{(x-c)^2+y^2}$$

The eccentricity of any ellipse is defined as $\frac{c}{a}$.

We can place a line at $x = \frac{a^2}{c}$. We can see that the result will be that this vertical line (LM in figure 3) will be our directrix. So this is the mathematical formula for the line we define as the directrix. Inspection of our figure shows why this is so.

Note that the distance PQ is equal to $\frac{a^2}{c} - x$. If you have trouble visualizing this, note that x is the horizontal

$\frac{a^2}{c}$

distance from the center of the ellipse to the x value of the position on the ellipse. But the total distance from the center of the ellipse to the directirx is $\frac{a^2}{c}$ since that is the x value for any point on the directrix line. In fact, that is exactly why we purposely place the directix at an x value of

Notice then that in the equation:

$$\frac{c}{a}\left(\frac{a^2}{c} - x\right) = \sqrt{(x-c)^2 + y^2}$$

the distance PF_2 from the position on the ellipse to the focus is:

$$\sqrt{(x-c)^2 + y^2}$$

and PQ is equal to:

$$\left(\frac{a^2}{c} - x\right)$$

Then we see that our equation really states:

PF_2 is equal to eccentricity, e, times PQ

This satisfies mathematically the requirement that the distance from the position on the ellipse to the focus is a fixed proportion to the distance from that position to a straight line. Hence, we have demonstrated the ellipse according to our second definition. We can proceed to use the relationships that we found in this chapter to define the position of the planets in polar coordinates in the next chapter. We want to be able to sue polar coordinates to locate the planet befcause these coordinates tell us where the planet is in degrees along its elliptical orbit. There will be situations wherein the Energy Equation will tell us where to find the planet along the ellipse if we are given the planet's position relative to the Sun and its total velocity.

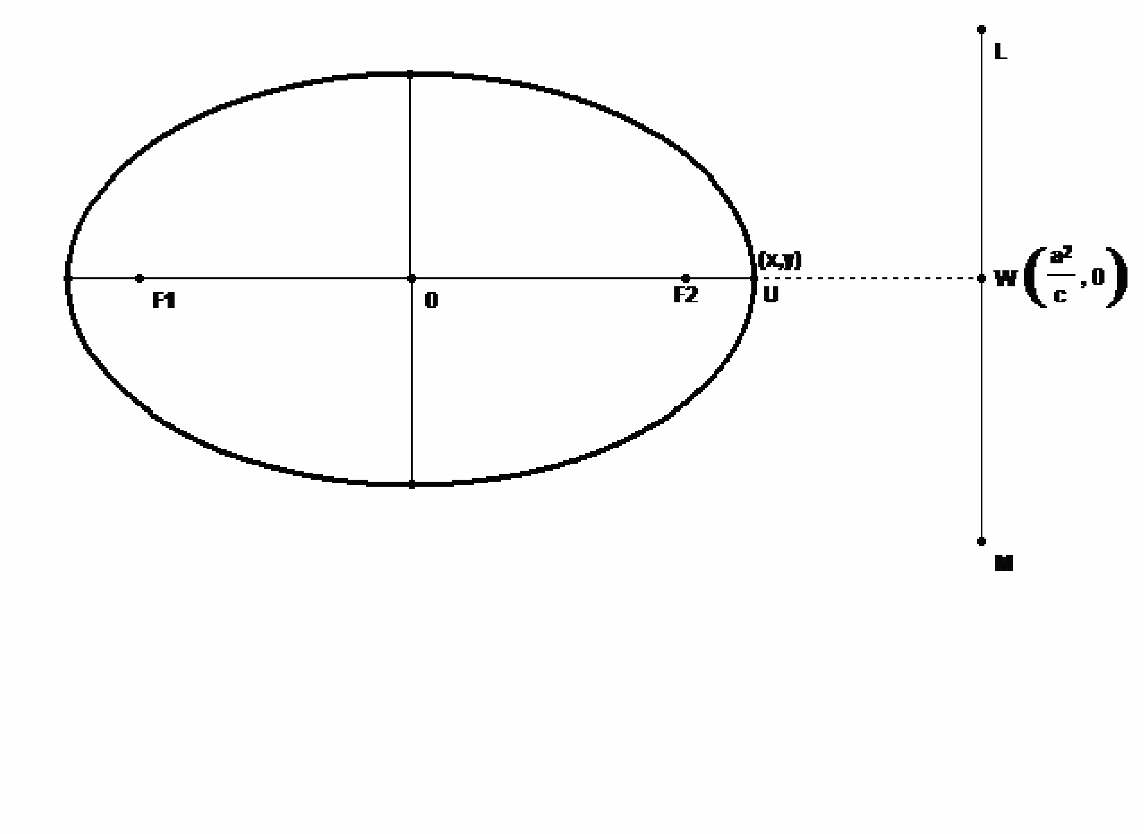


Figure 2 The planet is at perihelion. The value of x at the planet is by definition, *x*. The value of x at w is $\frac{a^2}{c}$. So the distance between *U* at perihelion and the directrix at *W* is $\left(\frac{a^2}{c}-x\right)$.

So the distance from the center of the ellipse to the directrix is:

$a+\left(\frac{a^2}{c}-x\right)=a+\left(\frac{a^2}{c}-a\right)$ since x is equal to a at perihelion.

Now simplify:

$$a+\left(\frac{a^2}{c}\right)-a=\frac{a^2}{c}=\frac{a}{c}a=\frac{1}{e}a=\frac{a}{e}$$

So the distance from the center of an ellipse to its directrix is equal to the length of the semimajor axis divided by the eccentricity of the ellipse.

Also notice that the distance from the perihelion of the ellipse to the directrix is equal to the distance from the center of the ellipse to the directrix minus the length of the semimajor axis. So:

$\frac{a}{e}-a=$the distance from the ellipse at perihelion to the directrix.

The relationships that have been demonstrated in this chapter will be useful in the next chapter where orbits are described in terms of polar coordinates.