Energy Equation Derived

It seems that in Chapter 33 the *a priori* proof of the Third Law would have been a good place to end this book. After all, by Chapter 33, the elliptical nature of orbits, the Inverse Square Law of Force, and the Third Law were sequentially shown to be natural and valid via *a priori* methods - thus accomplishing the primary aim of the book. But then in the four chapters leading up to this one, instead of halting, there is a shift in emphasis, still in *a priori* mode, to the derivation of a tool that allows the prediction of orbits and the analysis of what is happening to the planet at any given moment. In other words there is a shift from descriptive mode to analytical mode. The four previous chapters culminate here in the *a priori* proof of the Energy Equation, the relevant analytical tool. Specifically, it is shown in these five chapters that contained in the hodograph is a right triangle representing a relationship between total velocity and tangential velocity in terms of two variables - the distance to the Sun and the length of the semimajor axis. It is these relationships, inherent in the right triangle of the hodograph, that give us the Energy Equation.- a useful tool. And so, instead of simply ending at Chapter 33, this book will end with the application of the Energy Equation to various orbital situations in Chapter 42 and with a subsequent philosophical statement in chapter 43.

In this chapter we will tie together concepts to build the Energy Equation. We will allow the empirical finding of Galileo that unequal masses fall to the ground at the same rate but otherwise we will have gained this all from *a priori* methods. We have Galileo to thank, in that sense, for giving us the term GM to use in order to turn many of our proportions into equations.
We would have been valid without it but many of our equations including the Energy Equation that we will derive in this chapter would have remained as a proportion instead of an equation.

This chapter's theme contrasts with the conventional Energy Equation, found in texts related to celestial mechanics, which analyses the energy of a planet in terms of its kinetic energy and its negative potential energy. This requires wrestling with the concept of negative potential energy related to free fall from the distance infinity- a concept and situation I have difficulty envisioning.

Let's back up to Chapter 22 where we demonstrated the formula for force, having learned about force from the hodograph, $F = \frac{GMm}{R^2}$. We were able to get to $F \propto \frac{1}{R^2}$ using only *a priori* methods but turned that proportion into the equation using Galileo's empirical finding.

In Chapters 23 and 24 we used the force equation to derive the formula for the velocity of a planet in a circular orbit. Galileo's empiricism led to the transformation of the finding in the context of learning how to scale circular hodographs, of the proportion $V_{circle} \propto \frac{1}{\sqrt{R}}$ into the equation $V_{circle} = \frac{\sqrt{GM}}{\sqrt{R}} = \sqrt{\frac{GM}{R}}$. In this chapter we will use the equation for the velocity of a planet in a circular orbit for the demonstration of our Energy Equation.

Before proceeding, note that in Chapter 35 we showed via scaling methods for hodographs that escape velocity for an orbit when the planet is at a certain distance from the Sun is $\sqrt{2}$ times greater than the velocity of a planet in a circular orbit at that same distance: $V_{escape} = \sqrt{2} \times V_{circle}$.

Plugging in the equation for the velocity of a planet in a circular orbit :

$$V_{escape} = \sqrt{2} \times V_{circle} = \sqrt{2} \times \sqrt{\frac{GM}{R}}$$

Now let's square the above equation. The reason will become evident:

$$V^2{}_{escape} = \frac{2GM}{R}$$

This has a familiar appearance. We saw a similar mathematical terms in Chapter 38 pertaining to the total velocity squared.

$$V^2{}_3 = GM\left(\frac{2}{R} - \frac{1}{a}\right)$$

Multiply the above equation into separate terms:

$$V^2{}_3 = \frac{2GM}{R} - \frac{GM}{a}$$

Now look at what happens when we subtract total velocity squared from escape velocity squared. The result is a constant:

$$V^2{}_{escape} - V^2{}_{total} = \frac{2GM}{R} - \left(\frac{2GM}{R} - \frac{GM}{a}\right) = \frac{2GM}{R} - \frac{2GM}{R} + \frac{GM}{a} = \frac{GM}{a}$$

Abbreviated fashion, recalling that we refer to total velocity on the hodograph as V_3:

$$V^2{}_{esc} - V^2{}_3 = \frac{GM}{a}$$

This is our Energy Equation for elliptical orbits. It is a proud achievement using *a priori* methods. We would have arrived at the proportion, $V^2{}_{esc} - V^2{}_3 \propto \frac{1}{a}$ without using any empirical findings but allowing for the ground based observations of Galileo, we have obtained a full equation.

Now what is the meaning of this Energy Equation? The answer lies in the concept of kinetic energy , the energy of motion, which is traditionally expressed in terms of mass and velocity squared. In fact, as a matter of review, the standard formula in texts for kinetic energy is $\frac{1}{2}mv^2$. Now the mass of our planet does not change as we compare its present total velocity to its theoretical escape velocity and so we can ignore mass in the kinetic energy formula when we evaluate the difference between the two energy states- present versus theoretical escape. And we could fairly divide the Energy Equation by 2 to make the terms resemble kinetic energy formula more closely if we want to but that is not necessary.

Simply knowing that we are dealing with energy since our terms are velocity squared tells us the meaning of our Energy Equation. The Energy Equation states that for every instant and every position of the planet in its orbit the difference between the energy of motion required for escape and the energy of motion actually present is a constant. In other words the planet is always lacking a certain amount of energy that would allow it to escape no matter where it is in its orbit. When the planet is close to the Sun it is moving fast and has a lot of energy of motion to propel it far away form the Sun, but not enough to escape. When the planet is far from the Sun it has gained a lot of distance toward the feat of escape but has slowed down and has very little energy left to propel it further away. In both positions, the planet obeys the Energy Equation exactly.

Recall that earlier in this chapter we showed escape velocity to be $\sqrt{2}$ times the velocity of a planet in a circular orbit. And if we square that equation we get $V^2{}_{esc} = 2V^2{}_{circle}$. In other words the escape velocity squared is equal to twice the square of velocity of the planet in a circular orbit , both velocities pertaining to initial conditions at a specified radius to the Sun.

So we could further embellish upon the Energy Equation to say that it states that twice the circular velocity squared minus the actual velocity squared is always a constant. It is only mentioned here in terms of interest.

The importance of the Energy Equation goes well beyond its meaning. Since we have the constant GM in the numerator on the left side of the equation , we can predict what effect changes in velocity of our orbiting planet or spaceship would have on the denominator a. In other words, we have acquired a tool that will tell us what the new semimajor axis will be when we change the planet or spaceship velocity. This is an important tool in navigation from one orbit to the next desired orbit for space travel.

We can further put it to use in predicting things like how far a planet would travel at any point in its orbit if the velocity were turned directly away from the Sun and - we can show using logic that it would fly away as far as the distance $2a$ and then fall back to crash into the Sun.

Another use would be to predict, using extremely eccentric orbits how long it would take an object to free fall towards a central body form an extreme distance in a gravitational field where the force varies inversely with the square of the distance - an otherwise difficult calculation.