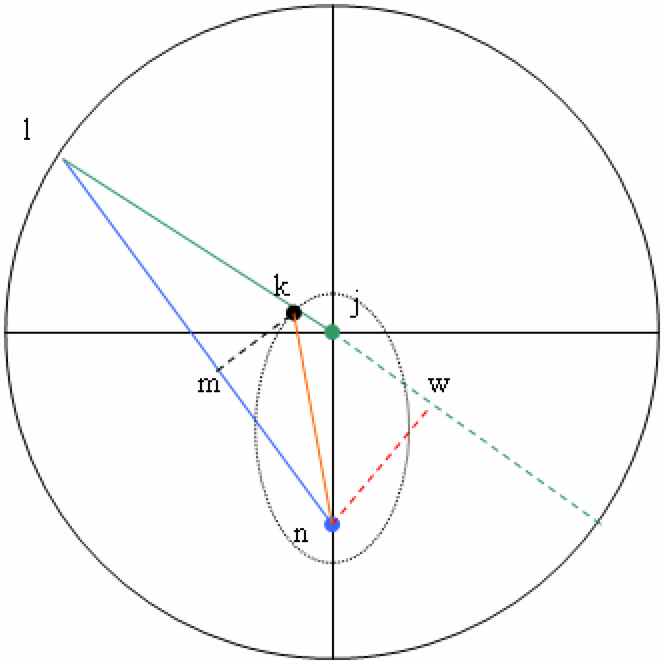
Total Velocity Squared

We are headed toward developing the Energy Equation for elliptical orbits. One of its key components, the square of the total velocity, can be made evident on the hodograph velocity diagram by noting the presence of the "small proportion triangle" that was described in Chapter 8. In a Chapter 39 we will see that velocity squared is significant because it determines the energy of a moving body.

Let's review some relevant aspects of the hodograph diagram.



In the figure above the dotted elliptical orbit is traced on the hodograph velocity diagram. The Sun is at *j* the first focus. The second focus of the ellipse is at *n* . The planet is at position *k*. The Inverse Proportion Machine is on the hodograph. Its segments are $\overline{nj}$ and $\overline{jl}$. The segment $\overline{jl}$ would spin around the tip of the segment $\overline{nj}$ to generate the hodograph velocity diagram. The segment $\overline{mk}$ is perpendicular to the segment $\overline{nl}$. The segment $\overline{wn}$ is perpendicular to the segment $\overline{wl}$. The segment $\overline{wl}$ represents the tangential velocity of the planet when it is at position k. The segment $\overline{wn}$ represents the radial velocity of the planet when it is at position k. The segment $\overline{nl}$ represents the total velocity of the planet when it is at position k. The segment $\overline{nk}$ represents one of the legs of the string and tack ellipse and the other leg is the segment $\overline{kj}$. Recall that since $\overline{nk} = \overline{kl}$ we know that $\overline{nk} + \overline{jk}$ is equal to the radius of the velocity circle $\overline{jl}$. Recall that we call this radius of the velocity circle V_2. We know that the radius of the velocity circle is equal to the length of $2a$. In other words $V_2 = 2a$ in terms of size. We showed in a previous chapter that the legs of the string and tack ellipse sum to $2a$.

Consider the two legs of the string and tack ellipse represented by the segments $\overline{nk}$ and $\overline{kj}$. Lets designate the legs of an ellipse to be L_1 and L_2. Let the segment from the second focus to the planet , $\overline{nk}$, be L_1. Since the sum of the two legs is $2a$, we can express L_1 as a fraction of the total length of the two legs of the ellipse as expressed in the fraction,

$\frac{L_1}{2a}$. As the position of the planet changes, and L_1 changes, the value of the fraction changes. Notice that the other leg of the ellipse, L_2, represented above by $\overline{kj}$, is also the representation of the distance to the Sun which we will call r in this chapter.

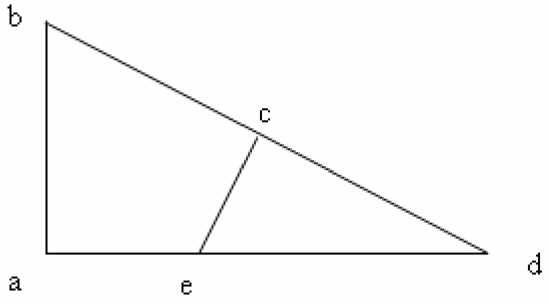
Since the legs L_1 and L_2 of the ellipse sum to $2a$:

$$L_1 = 2a - r$$

and so:

$$\frac{L_1}{2a} = \frac{2a - r}{2a}$$

Recall that in Chapter 8 we studied the Small Proportion Triangle.



We defined that $\overline{ba}$ is perpendicular to $\overline{ad}$ and $\overline{ce}$ is perpendicular to $\overline{ad}$.

We also defined that c is at the midpoint of $\overline{bd}$.

We showed that $\left(\overline{cd}\right)^2 = 2\overline{ad}\left(\overline{ed}\right)$

Look at the hodograph above to see the Small Proportion Triangle, Δwnl.

By applying the relationship between the segments of a Small Proportion Machine we see that in the hodograph above:

$$\left(\overline{nl}\right)^2 = 2\overline{wl}\left(\overline{kl}\right)$$

Also recall that $\overline{kl} = \overline{nk}$ and $\overline{nk}$ represents the length L_1 which in turn we showed is equal to $2a - r$. Also note that $\overline{wl}$ represents tangential velocity when the planet is at position p. Finally note that $\overline{nl}$ represents total velocity.

Let's designate total velocity as V_3 for simplicity in writing formulae.

Then our equation above becomes:

$$V_3{}^2 = 2V_t L_1$$

Now we wish to express L_1 instead in terms of a fraction of V_2, the radius of the velocity diagram.

Note that by inspection of the diagram the segment $\overline{kl}$ can be defined by the fraction of V_2 that it occupies. In other words, some fraction times V_2 is equal to segment $\overline{kl}$. Note that V_2 is equal to $2a$ in size. Note that L_1 is the same fraction of 2a as the fraction that $\overline{kl}$ is of V_2 so that :

$$\frac{L_1}{2a} = \frac{\overline{kl}}{V_2}$$

Now since $\frac{\overline{kl}}{V_2}$, the fraction, times the whole of V_2 is obviously equal to $\overline{kl}$:

$$\frac{\overline{kl}}{V_2} \times V_2 = \overline{kl}$$

we see that there is a way to express L_1 as a fraction of V_2 multiplied by the whole of V_2.

We can similarly state:

$\frac{L_1}{2a} \times 2a = L_1$ and being that V_2 is equal in size to $2a$ we can substitute:

$$\frac{L_1}{2a} \times V_2 = L_1$$

And so we can further substitute $2a-r$ for L_1 so that we can say:

$\frac{2a-r}{2a}\times V_2 = L_1$ and we can substitute this expression into the equation that the Small Proportion Machine revealed to us:

$V^2{}_3 = 2V_t L_1$ so:

$$V^2{}_3 = 2V_t\left(\frac{2a-r}{2a}\right)V_2 = V_t\left(\frac{2a-r}{a}\right)V_2 = V_t\left(2-\frac{r}{a}\right)V_2 \text{ .}$$

So we have an *a priori* relationship between total velocity, tangential velocity, semimajor axis, and the radius of the velocity circle of the hodograph .

Now let's recall that the areal velocity is:

$h = V_t \times r$ as we showed in our wedge of area swept in a unit of time in Chapter 25.

So we can get areal velocity into our equation above by rearranging the equation for areal velocity and plugging in for tangential velocity:

$$V^2{}_3 = \frac{h}{r}\left(2 - \frac{r}{a}\right)V_2 = hV_2\left(\frac{2}{r} - \frac{1}{a}\right)$$

Now recall that in a Chapter 37 we showed that for the velocity circle and hodograph:

$$GM = hV_2$$

(We used Gallileo's empirical finding that unequal masses fall to the ground at the same rate to get at the above term GM but otherwise we are still on an *a priori* path. We have used no empirical astronomical data or observations.)

Substituting to get GM into the equation we get:

$$V^2{}_3 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

And this is a recognizable relationship derived in previous texts. Without Gallileo's empircial finding we would have been able to arrive at the proportion in *a priori* fashion:

$$V^2{}_3 \propto \left(\frac{2}{r} - \frac{1}{a}\right)$$

In the next chapter we will use the above relationship to build further the Energy Equation for elliptical orbits.