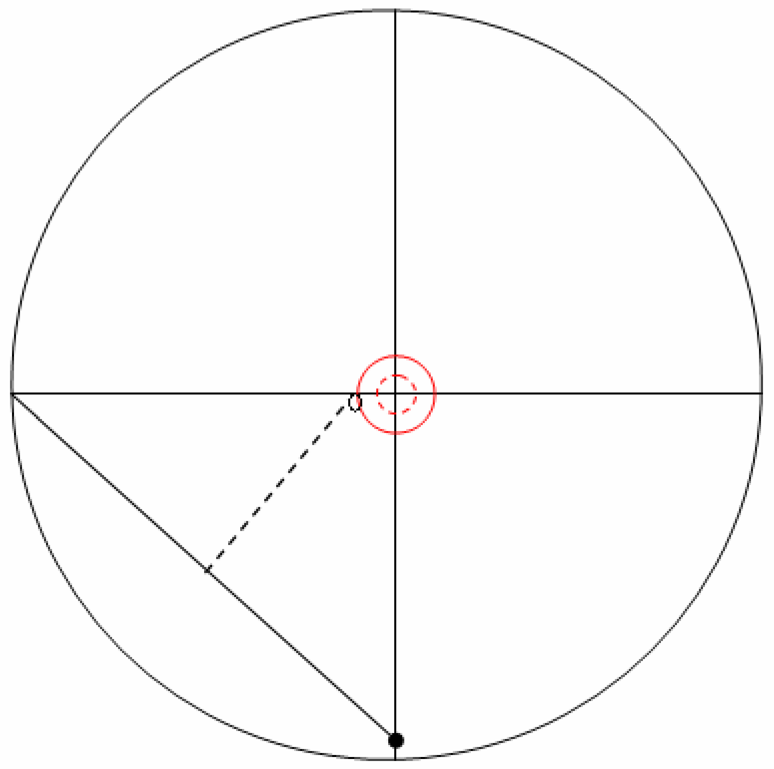
# To Escape Velocity

In this chapter we will show the relationship between circular velocity and escape velocity for an orbit around a central body. We will use the scaling method that we devised for hodographs.

Imagine that we have a planet in a circular orbit around the Sun. The eccentricity of the circular orbit is zero. We want to know how fast the planet would have to travel in order to escape from the Sun.

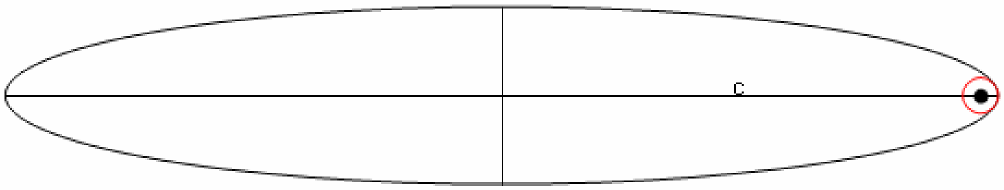
As an orbit becomes more eccentric, it becomes closer to an escape path, and closer to a straight line trajectory away from the Sun. The real key is that the velocity increases enough so that the planet keeps going and never returns to the Sun. As eccentricity increases to values close to 1, the length of the semimajor axis approaches infinity but we don't need to consider the infinite semimajor axis. We only need to see how increasing eccentricity and velocity interact. We can use our scaling methods to explore this interaction. We will see a mathematical trend developing as eccentricity increases which reveals the relationship that the escape velocity at any given radius is $\sqrt{2}$ times the velocity of the planet in its circular orbit.

In our scenario, we start with a planet at a given position in a circular orbit and then boost the velocity so as to shoot the planet from the given position into an extremely eccentric orbit of exceedingly long semimajor axis. For example let the eccentricity increase to 0.99. We could choose 0.99999999 if we wanted to but let's keep the numbers manageable. Besides, we are going to discover a pattern which will make such long digit numbers unnecessary. We can draw our shooting planet and the initial circling planet on the same hodograph velocity diagram. That way we will know how to properly scale their velocities.



Inspect the figure above to see that two orbits are portrayed on it. In red is a hodograph for the planet in its circular orbit, the dashed inner circle representing the actual orbit and the outer solid red circle representing the velocity circle. The circular velocity arrow , which we will designate $V_c$ would be twice the length of the orbit of the radius as would be dictated by the Inverse Proportion Machine. The shooting planet with a highly eccentric orbit is represented in black as it is at position $p$. We assigned the eccentricity to be 0.99. In

Chapter 2 we showed that eccentricity $e = \frac{c}{a}$. Let the original radius of the circular orbit be one unit of distance. We shoot our planet out from its circular orbit into the orbit of eccentricity equal to 0.99. In that case the semimajor axis will be 100 distance units since $e = \frac{c}{a} = .99$ so that when our planet flies outward, it enters an elliptical orbit for which the perihelion distance is equal to 1, the radius of the initial circular orbit, and $c$ becomes 99 distance units to satisfy the condition that eccentricity is .99.



The original circular orbit is shown in red above and the theoretical orbit of eccentricity equal to .99 as the planet is shot far outward as shown in black. Notice how relatively large $c$ becomes and notice that $a = 100$ for the elliptical orbit.

Let’s examine the “almost escape” orbit at perihelion:

Looking back at the hodographs for the two orbits at the top of this chapter we can scale the velocities using our scaling method. By inspection, since we set the circular orbit radius to equal 1 and since the elliptical orbit has a semimajor axis equal to 100, we see that the perihelion “almost escape” velocity is four times the length of the semimajor axis since the radius of a velocity circle is twice its semimajor axis length and the velocity arrow is the entire diameter of the velocity circle for the “almost escape” velocity orbit. So we assign the perihelion “almost escape” velocity to literally appear to be 400 units in size. The velocity arrow for the initial circular orbit is twice the semimajor axis as is the case for all circular orbits and so it measures 2 units in size. So the perihelion “almost escape” velocity orbit has a velocity 200 times as large as the initial circular orbit according to the literal sizes of the velocity arrows on the hodograph before scaling.

Our scaling method tells us that velocity is proportional to $\frac{1}{a} \times \frac{1}{\sqrt{p}}$. Now the first term tells us to decrease the almost escape velocity arrow by multiplying it by $\frac{a_{circle}}{a_{ellipse}} = \frac{1}{100}$. For the next term, we must calculate $p$ for the "almost escape" ellipse. In a Chapter 2 we saw that $p = a\left(1-e^2\right)$.

So for the "almost escape" ellipse $p = 100\left(1-.99^2\right) = 100(1-.98) = 100(.02) = 2$. For the circular orbit $p$ is equal to the radius of the initial circular orbit which is equal to 1. We have to scale to $\frac{1}{\sqrt{p}}$.

So since $p$ is twice as large for the "almost escape" ellipse as it is for the initial circular orbit we must further decrease the velocity arrow of the almost escape ellipse by multiplying it by $\frac{1}{\sqrt{p}} = \frac{1}{\sqrt{2}}$.

So the final true "almost escape" velocity arrow size will be the literal "almost escape" arrow size ( of 200 times the size of the circular velocity arrow) on the hodograph times the first scaling factor $\frac{1}{100}$ and then times the second scaling factor $\frac{1}{\sqrt{2}}$:

$$200 \times \frac{1}{100} \times \frac{1}{\sqrt{2}} = 2 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

In other words the “almost escape” velocity arrow is $\sqrt{2}$ times the arrow for the initial circular orbit.

Now it turns out mathematically that as we use the same theoretical situation but keep increasing eccentricity to .99999999 and beyond, the mathematical terms for eccentricity and $p$ and the increasing value of $a$ balance out so that the final proportion between the velocity arrows remains at exactly $\sqrt{2}$.

And so we can state the important relationship between escape velocity and circular velocity for a planet at a given distance from the Sun. Escape velocity is equal to circular velocity times $\sqrt{2}$. Note that we have essentially demonstrated that we can start with a circular orbit at a given distance to the Sun. Then we let that distance from the Sun be perihelion for an “almost escape” trajectory to derive:

$$V_{esc} = \sqrt{2}V_{circle}$$

In the next chapters, we will apply this relationship to proportions of specific segments in the hodograph in order to derive the Energy Equation for elliptical orbits.