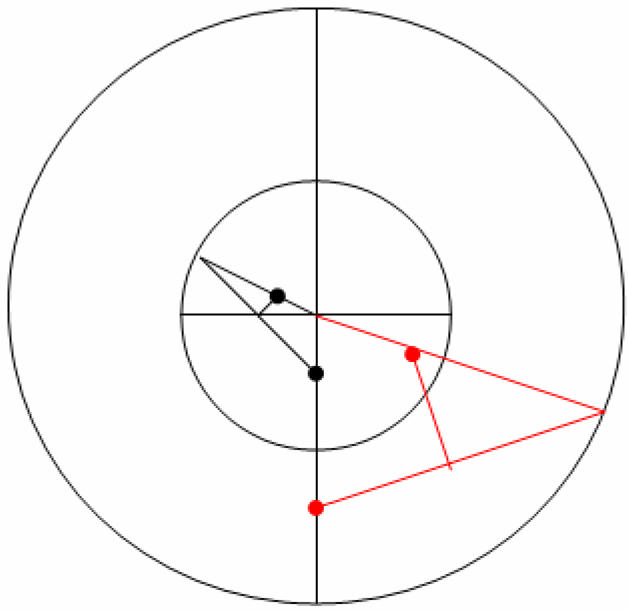
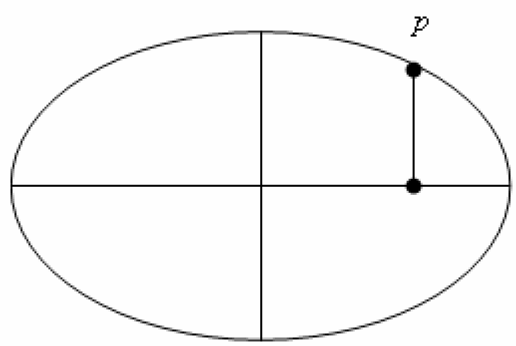
Scale Various Axes

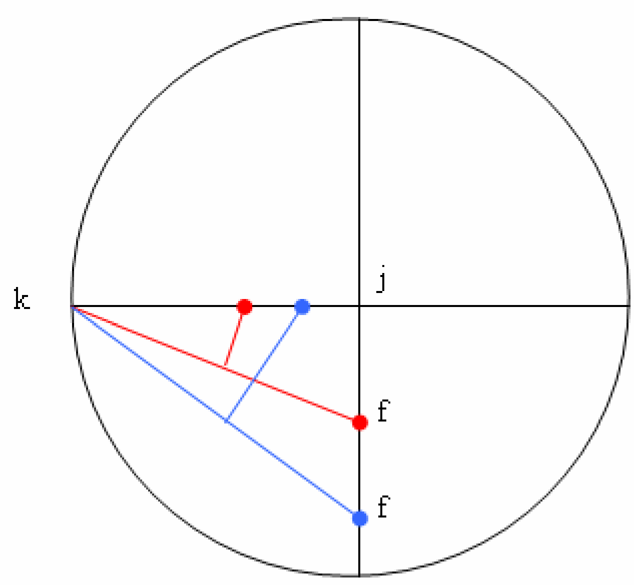
So far we know how to scale for circular orbits in the same solar system and elliptical orbits that have the same semimajor axis in the same solar system. In the next two chapters we will show the more general method of scaling so that we know how to scale any two orbits in the same solar system. In other words we will expand our knowledge so that we will know how to deal with scaling various size elliptical orbits in the same solar system even when their semimajor axes differ in length. In that case we will have different size velocity circles drawn on the same page. For example, a new species of hodograph would look like this:



We will first examine, within a solar system, planets at the top of the semilatus rectum of elliptical orbits of equal semimajor axis length. We say that the planet is at position p when it is at the top of the semilatus rectum.



The hodograph for two elliptical orbits is represented on the same velocity hodograph below:



The planets of the two orbits must have the same length semimajor axis since they are drawn on the same velocity circle. (Recall that the radius of a velocity circle is equal to twice the length of the semimajor axis.) The red and blue lines represent the planets when they are at position p . The second foci for both planets are labelled f. The segment $\overline{jk}$ represents the tangential velocity of both of the planets when they are at position p. The hodograph if taken literally would mislead us to state that the tangential velocity of the two planets is the same. But this is not truly the case as we will see. And so we will see how to scale the tangential velocity arrow for these planets at position p.

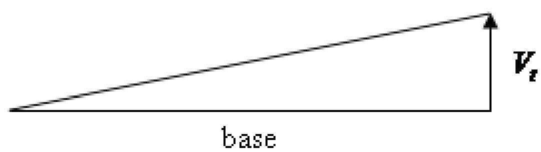
We learned in the Chapter 33 that planets of equal semimajor axis have the same orbital period and that the period is related to semimajor axis:

$$T \propto a^{\frac{3}{2}}.$$

Let's apply that knowledge to the hodograph above. The period can also be defined as twice the area swept divided by the areal velocity as we saw in Chapter 33:

$T = \frac{2\pi ab}{h}$

Now h is the twice the area of wedge swept or once times the product of the base times the tangential velocity in a single tiny unit of time:



Now for the planet at position p, the base is the distance p, and the far side is numerically equal to the tangential velocity V_t since our time unit equals 1. We will label, for convenience, the tangential velocity of the planet at position p, calling it V_{t90} since it is the tangential velocity of the planet when the planet is at an angle of 90 degrees relative to the perihelion position of the semimajor axis of the elliptical orbit.

Since planets within an orbit sweep equal areas in equal times, the areal velocity is equal at any position along the orbit. Areal velocity, h, is represented by doubling the triangular wedge above. At position p for any planet in any solar system:

$$h = V_{t90} \times p$$

So $T = \frac{2\pi ab}{h} = \frac{2\pi ab}{V_{t90} \times p} \propto \frac{ab}{V_{t90} \times p}$

And we also know that for all planets of various semimajor axes within a solar system :

$T \propto a^{\frac{3}{2}}$ so that we can combine the proportions to say:

$$a^{\frac{3}{2}} \propto \frac{ab}{V_{t90} \times p} \propto \frac{ab}{V_{t90}}\left(\frac{1}{p}\right) \propto \frac{ab}{V_{t90}}\left(\frac{a}{b^2}\right) \propto \frac{a^2}{V_{t90} \times b}$$

So restating the above line:

$$a^{\frac{3}{2}} \propto \frac{a^2}{V_{t90} \times b}$$

And solving for V_{t90}:

$V_{t90} \propto \frac{a^2}{a^{\frac{3}{2}} \times b} \propto \frac{\sqrt{a}}{b} \propto \frac{1}{\sqrt{p}}$. Note we used $\sqrt{p} = \frac{b}{\sqrt{a}}$ . .

So $V_{t90} \propto \frac{1}{\sqrt{p}}$.

This is the reality for planets at position p on different elliptical orbits of various sized semimajor axes within a solar system. Note that this finding is not limited to orbits of equal semimajor axes. But the hodograph above, where the two planets have the same semimajor axis, told us to expect that the V_{t90} would be equal for all these planets. And so that hodograph will be instructive. It tells us that when the semimajor axis of elliptical orbits is equal within a solar system that the true scaling factor is $\frac{1}{\sqrt{p}}$, not simply $\frac{1}{b}$.

So we learn that we must scale velocity to $\frac{1}{\sqrt{p}}$ for these planets of equal semimajor axes within a solar system. This is the definitive scaling method. In Chapter 33 we showed that for orbits of equal semimajor axes within a solar system that velocity scales to $\frac{1}{b}$. Is there a contradiction? No. The semimajor axes are equal so :

If a is held constant then:

$\frac{1}{\sqrt{p}} \propto \frac{1}{b}$ since $\frac{1}{\sqrt{p}} = \frac{\sqrt{a}}{b} \propto \frac{1}{b}$ since a is a constant. And as a corollary, the proportion $T \propto a^{\frac{3}{2}}$ also still would hold true even though we used $\frac{1}{b}$ instead of $\frac{1}{\sqrt{p}}$ since the constant $\sqrt{a}$ drops out in each step of the proportion's derivation.

So to restate what the past several chapters and this chapter have found is:

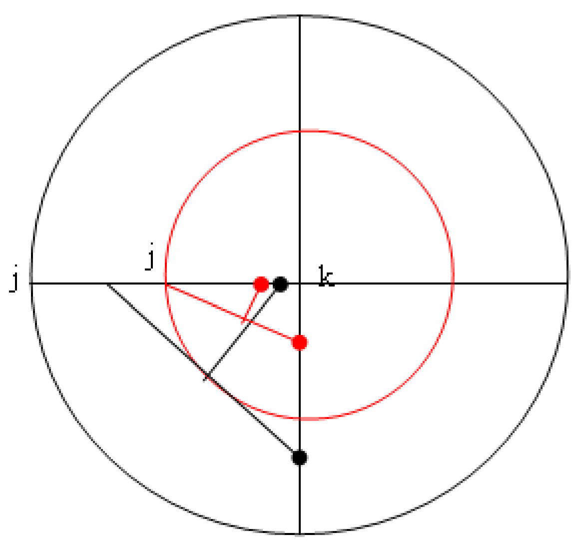
Within a solar system for orbits of equal semimajor axes, we studied the planets at position b to learn that hodograph velocity arrows must be scaled to $\frac{1}{b}$.

We learned that scaling as such led to the proportion $T \propto a^{\frac{3}{2}}$.

Then we studied the planets at position p using the proportion $T \propto a^{\frac{3}{2}}$ to learn the refinement that true V_{t90} velocity, even for orbits of different semimajor axis, must be proportional to $\frac{1}{\sqrt{p}}$.

Why is this important? It is critically important because we will be able to use reason to determine a scaling method that correctly yields the proportion, $V_{t90} \propto \frac{1}{\sqrt{p}}$, for all planets in a solar system. This will allow us to properly scale hodographs when the semimajor axis, a, varies.

Reason tells us that we can inspect two hodograph velocity circles to obtain the correct universal scaling method for planets within a solar system.



Note that there are two orbits represented above, one in black and one in red. The planets are each represented at position p. The black orbit has a larger semimajor axis than the red orbit. The eccentricity of the black orbit is larger than the eccentricity of the red orbit so the black planet is actually closer to the Sun than the red planet when they are both at position p in their orbits. Note that for each planet V_{t90} is represented by the segment $\overline{kj}$.

We know that true V_{t90} must be proportional to $\frac{1}{\sqrt{p}}$.

By inspection to accomplish the true proportion $V_{t90} \propto \frac{1}{\sqrt{p}}$ we see we can do it in two steps.

The first step is to equalize the radii of the red and black velocity circles. We showed in Chapter 27 that the radii of the velocity circles are proportional to a by properties of the Inverse Proportion Machine. So we use the scaling factor $\frac{1}{a}$ to equalize the velocity circle radii in the first step. For example if the black radius is 10 and the red radius is 4 we scale down the black radius by multiplying it by $\frac{4}{10}$. This is mathematically equivalent to scaling to $\frac{1}{a}$. By inspection, equalizing the radii equalizes the segments $\overline{kj}$ which represent V_{t90} for each orbit.

By logic, the only remaining step is to get these equalized segments to be proportional to $\frac{1}{\sqrt{p}}$ and that is self evident. Since the segment has been equalized, the way to adjust it so that it is proportional to $\frac{1}{\sqrt{p}}$ is simply to assign the scaling factor $\frac{1}{\sqrt{p}}$ as the second step.

To summarize, we find in our two step method that the way to scale the hodograph in general, for orbits within a solar system, is to scale velocity arrows proportional to $\frac{1}{a} \times \frac{1}{\sqrt{p}}$.

In a few sentences let's justify our scaling method so that we can visualize it and know that it is valid:

Since true V_{t90} must be proportional to $\frac{1}{\sqrt{p}}$ for all planets in a solar system, the correct scaling method will result in the V_{t90} velocity arrows, after scaling, being proportional to $\frac{1}{\sqrt{p}}$. This is achieved in two step fashion, by first applying the ratio, $\frac{1}{a}$, that geometrically and conveniently equalizes the length of the V_{t90} velocity arrows and then subsequently applying the desired algebraic ratio, $\frac{1}{\sqrt{p}}$, that we know actually governs the relationship between V_{t90} and p.

In the next chapter we will apply our scaling method to learn about escape velocity.