The Periods

In Chapter 24 Kepler's Third Law for circular orbits was derived in an *a priori* fashion (allowing Galileo's empirical findings to supply the Gravitational Constant in order to convert the *a priori* proportion into its equation form). It has been a mathematical adventure since then; In this chapter it we see that the adventure has been preparing us to do the following:. We will apply our newly acquired scaling method to derive the Third Law for elliptical orbits in *a priori* fashion. We will demonstrate why $T \propto a^{\frac{3}{2}}$ where a is the semimajor axis of the ellipse.

In Chapter 29 we studied a hodograph velocity diagram with two orbits drawn on it. Its circular orbit and elliptical orbit have the same semimajor axis - for the circular orbit this is the radius.
We observed that the hodograph implied that $V_t \propto b^2$ but then we studied the force acting upon planets at position b and learned that we must scale velocity to $\frac{1}{b}$. We used logic to extend this scaling method to the hodograph that contains

two elliptical orbits that have the same semimajor axis but differ in eccentricity and thus semiminor axis. And now we can with good reward apply our scaling method to tangential velocity as follows:

$$V_t \propto b^2 \times \frac{1}{b} \propto \frac{b^2}{b} \propto b$$

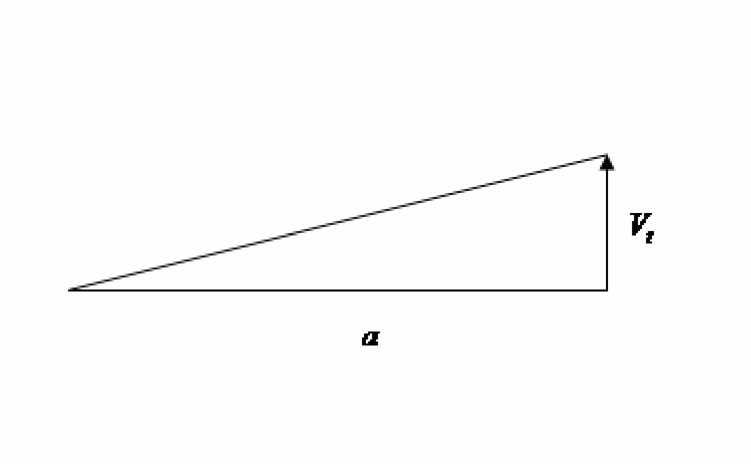
In other words, having learned how to scale the hodograph we see that it teaches us that when its orbits and have the same semimajor axis, the actual tangential velocity when the planet is at position b ,is proportional to b itself.

That is a phenomenal finding.

We will use it to find the period of the ellipses drawn on the same velocity hodograph. Notice that b is part of the formula we demonstrated for the area of an ellipse, πab. And recall that we showed in *a priori* fashion that equal areas are swept in equal times within orbits.

So look again at our elliptical orbits when the planets are at position b.

Within each elliptical orbit, equal areas are swept in equal times. So if we have the area swept in any time period we have the area swept per unit of time which holds true for any place along the orbit. Recall that in Chapter 25 we stated our convention to call h the areal velocity and to define it as twice the area swept per unit time. We showed in Chapter 27 that for all the orbits on the same hodograph the semimajor axes are equal. And when the planets of those orbits are at position b we know they are all at the same distance from the Sun since we showed in Chapter 2 that planets are at a distance equal to the semimajor axis when they are at position b. One half of the area swept at position b is equal to the radius to the Sun multiplied by the tangential velocity. The radius to the Sun is a, the semimajor axis, for all these planets:



So we see the wedge of area swept. We are reminded that it is a triangle and that it is half the product of the radius and the tangential velocity. We see that the triangle is half of the area represented in areal velocity and see therefore the relationship that areal velocity is twice the area swept in a unit of time.

Now we return to our planets of equal semimajor axes. At position b the radius of the wedge above is equal to a for all these planets. The far side of the wedge is proportional to the tangential velocity in the unit of time swept. We have learned that tangential velocity is proportional to b. Thus the wedge of area swept is proportional to b. In mathematical terms:

The areal velocity, twice the area swept in a unit of time is:

$$h = V_t \times a \propto b \times a$$

And since a is constant for our planets on the same hodograph diagram when they are at position b:

$$h \propto b$$

Now although we derived h for the planet when it is at position b we know that it holds for h anywhere along the orbit since equal areas are swept in equal times anywhere along the orbit.

Equal areas are swept in equal times within orbits so the entire time period of a complete orbit relates to areal velocity as follows:

Areal velocity, twice the area swept in a unit of time, is twice the elliptically shaped area swept during one orbit divided by the period of one orbit:

$$h = \frac{2\pi ab}{T}$$

Rearranging and stating as a proportion:

$$T \propto \frac{2\pi ab}{h}$$

Now we showed above by applying our scaling method that for our planets on the same hodograph diagram:

$$h \propto b$$

So $T \propto \frac{2\pi ab}{h} \propto \frac{2\pi ab}{b} \propto 2\pi a$

And since a, the semimajor axis is equal for all these planets on the same hodograph:

T is a equal for all these planets.

This is a monumental finding. It states that the planets in the same solar system that have the same semimajor axis have the same time period of orbit. Now recall that one of these planetary orbits is allowed to be circular. And that is the key to generalizing to the formula:

$$T \propto a^{\frac{3}{2}}$$

Within a solar system for all the possible orbits of a given semimajor axis, one of those possible orbits is the circular orbit of radius equal to a. Its period is identical to that of all the elliptical members of its family. In Chapter 24 we showed that for circular orbits within a solar system we know the proportion between radius for circular orbits and period:

$$T \propto R^{\frac{3}{2}}$$

So we have a proportion for circular orbits and periods. But the periods of elliptical orbits are equal to the periods of circular orbits when their semimajor axes equal the radii of those circular orbits. So the semimajor axes are related to the periods the same way that the radii of circular orbits relate to the periods. And so :

$$T \propto a^{\frac{3}{2}}$$

And we have arrived at Kepler's Third Law for elliptical orbits using *a priori* methods.