

## False Force at b

In Chapter 31 we saw that within a solar system, if the semimajor axis of an elliptical orbit is equal to the radius of a circular orbit, and if we examine the planet when it is at position  $b$ , for the ellipse, and anywhere on the circle, the following proportion seems to hold for the angle swept in tiny instants of time:

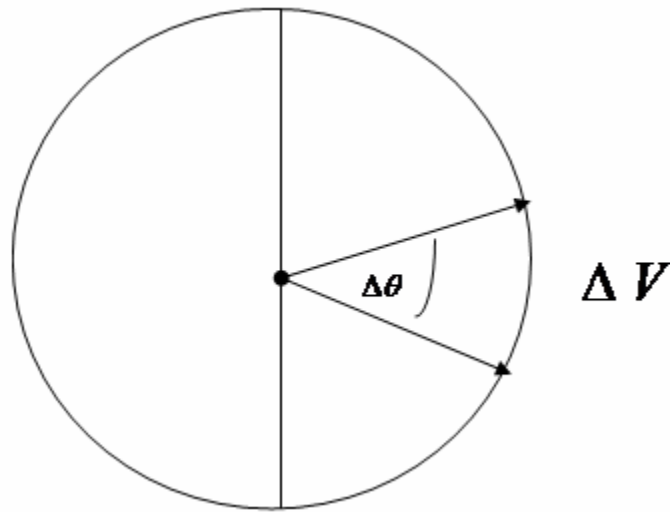
$$\theta_{circle} = \theta_{ellipse} \left( \frac{b^2_{circle}}{b^2_{ellipse}} \right)$$

(Note that the above equation could be algebraically manipulated so that it would also serve to compare angles swept for two different elliptical orbits of equal semimajor axis in the same solar system and that their proportion would be regulated by the ratio of the squares of their semiminor axes. But we do not need this comparison in order to proceed - especially since the equation above turns out to be false.)

Remember that the equation above will be shown to be false. But for now assume that it is correct since the hodograph tells us it is true.

Also recall from Chapter 20 that for any hodograph, Feynman and Goodstein and Goodstein showed us, and the Inverse Proportion Machine dictated *a priori* that:

$\frac{\Delta V}{\Delta \theta}$  is a constant as we travel around the circumference of the hodograph velocity diagram for orbits.



And now realize that the hodograph of the circular orbit and the elliptical orbit in the previous chapter are represented on the same hodograph velocity diagram. So if we take what we see literally and apply the relationship  $\frac{\Delta V}{\Delta \theta}$  is constant then we can, for the small angle swept in the tiny time interval, say that we can substitute  $\Delta V$  for

$\Delta\theta$  ,since they change in direct proportion to each other,  
and transform our equation into a proportion that states:

$$\Delta V_{circle} \propto \Delta V_{ellipse} \left( \frac{b^2_{circle}}{b^2_{ellipse}} \right)$$

Now realize that the above equation relates to what the  
hodograph is telling us to expect in a small time interval.  
And force is measured by the change in velocity divided by  
the change in time:

$$F = ma = m \frac{\Delta V}{\Delta t}$$

To continue in a *a priori* fashion we pause to state that we  
can fairly assume that our two planets have equal mass for  
the purposes of further analysis. All that we are deriving  
from the hodograph is still valid in these chapters about  
the planet at position  $b$ . But we could venture the same  
distance into empirical fashion as we did before by  
allowing Gallileo's finding that unequal masses fall at the  
same speed. This led to the formula  $F = \frac{GMm}{R^2}$  in Chapter

22. If we then equate  $F = \frac{GMm}{R^2} = m \frac{\Delta V}{\Delta t}$  we see the masses of the planet cancels out and actually plays no role in the way it moves.

So returning to our proportion above, notice that during the small time period we can find the force by looking at the  $\frac{\Delta V}{\Delta t}$  for each side of the proportion noting that the

time period is the same for both  $\Delta V_{circle}$  and  $\Delta V_{ellipse}$  :

$$\frac{\Delta V_{circle}}{\Delta t} \propto \frac{\Delta V_{ellipse}}{\Delta t} \left( \frac{b^2_{circle}}{b^2_{ellipse}} \right)$$

Now this implies that the force for the circular orbit is different from the force for the elliptical orbit by the square of the ratio of the semiminor axes -recalling that the semiminor axis of the circular orbit is equal to its radius.

But recall that the planet when it is at position  $b$  is always at the distance equal to the length of the semimajor axis. Our planets have the same semimajor axis  $a$  as we showed in Chapter 29 when we set up our hodograph. In

Chapter 21 we derived in a *a priori* fashion  $F \propto \frac{1}{R^2}$ . So we

see a paradox. The two planets are at the same distance from the Sun. The force on the planets must be the same. But the hodograph, if taken literally tells us that the force is different by the ratio:

$$\frac{b^2_{circle}}{b^2_{ellipse}}$$

In other words the hodograph tells us that the force on the two planets differs according to the square of the semiminor axes.

Therefore the hodograph must be scaled so that it will tell us instead that the force on our two planets is the same since they are at the same distance from the Sun. Insight gained from the experience of learning how to scale the hodograph for circular orbits tells us roughly that we can draw orbits themselves to proper scale on the same velocity diagram but it is the velocity arrows that need to be scaled to be larger or smaller. So the present situation

brings up the question: What change in scale for velocity will result in the force at position  $b$  being equal for our two planets?

We can get at the answer by studying what effect the change in scale of velocity will have on the factors that we used to determine the force. Velocity was used to determine first the angle swept by the planet in the instants that it is at position  $b$ . And then velocity was used again when the circumference of the velocity circle and  $\Delta V$  were studied. So changing the scale of velocity will have an effect twice as it influences what the hodograph truly represents regarding force at position  $b$ .

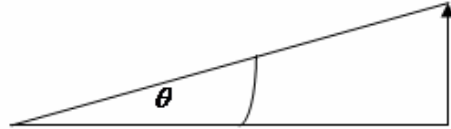
We are close to the answer. We can reach it with logic. We know that the force must be the same for our two planets. But the hodograph tells us that the orbit with the larger semiminor axis has force that is too large by a factor equal to the square of the ratio of the semiminor axes. So the semiminor axes must be used somehow to scale the velocity. Since scaling the velocity will have an effect twice, it will be an effect analogous to squaring the initial effect. For example if we scale during our first effect by multiplying something by 5 and then repeat the effect by increasing the first result by 5 we will have made the original amount increase 25 fold. Our

scaling will have to make the velocity smaller for the larger semiminor axis and will have to square to give a negating effect, (or in mathematical terms "cancel out the term") for the  $\frac{b^2_{circle}}{b^2_{ellipse}}$  expression . By inspection, this would result if velocity were scaled to  $\frac{1}{b}$ .

If that were the case the  $\frac{b^2_{circle}}{b^2_{ellipse}}$  term would disappear and the force on the two planets would adjust mathematically to be the same. And therefore, this is indeed the correct way to scale.

We can illustrate why this should be so by demonstrating the two effects of scaling velocity to  $\frac{1}{b}$ . As we stated above, changing the scale of velocity will have an effect in two places. First in determining  $\Delta\theta$  and next in determining  $\Delta V$ .

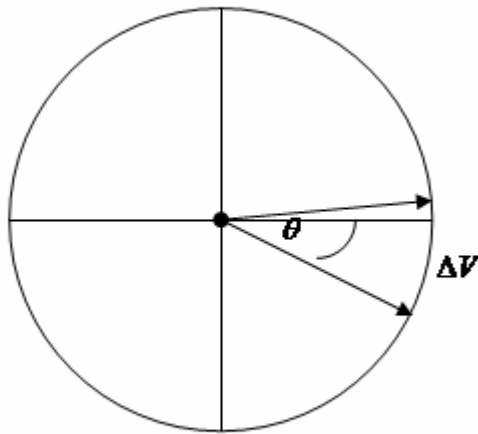
Recall the wedge that determines  $\Delta\theta$ :



$$\text{Distance} = V_t \times \text{time} \equiv V_t \Rightarrow V_t \times \frac{b_{\text{ellipse}}}{b_{\text{circle}}}$$

Above is the wedge whose far side is determined by tangential velocity in the tiny unit of time. We assigned the time unit the value of 1 so that the distance would be numerically equal to the tangential velocity. The wedge above is the one that represents the circular orbit . We scale the velocity to  $\frac{1}{b}$  by multiplying by  $\frac{b_{\text{ellipse}}}{b_{\text{circle}}}$  which makes the tangential velocity for the wedge of the circular orbit smaller. Now by scaling the velocity so that the tangential velocity becomes smaller has a direct effect on the length of the far side of the wedge which in turn has a direct effect by definition on the tangent of the angle swept and since the tangent of a tiny angle is equal to the angle itself when measured in radians the angle swept ill

be directly affected proportional to  $\frac{1}{b}$ . So the  $\Delta\theta$  is scaled to  $\frac{1}{b}$ . Now this  $\Delta\theta$  spans an arc along the velocity circle. This is the place where scaling has its second effect.



Since velocity scales to  $\frac{1}{b}$ , the units of velocity along the arc on the velocity circle are also changed in scale by the proportion  $\frac{1}{b}$ .

So not only is the angle  $\theta$  made smaller by scaling, but also the units of the velocity arc that  $\theta$  spans along the

velocity circle. So scaling velocity to  $\frac{1}{b}$  acts twice, in a sequence, so that the net result is that the change in velocity during the time that the angle  $\theta$  is swept scales to  $\frac{1}{b} \times \frac{1}{b} = \frac{1}{b^2}$ . And this scaling correction is what mathematically makes the force equal for our two planets by negating the  $\frac{b^2_{circle}}{b^2_{ellipse}}$  term in the equation for force written above for the two planets.

Now that we have a scaling factor to use for the hodograph of a circular and elliptical orbit on the same velocity diagram we have the key to the rest of the analysis. Knowing that velocity scales to  $\frac{1}{b}$  will teach us how to derive Kepler's Third Law for elliptical orbits and also teach us how to scale hodographs more generally, where the semimajor axes are not equal and the orbits are elliptical.