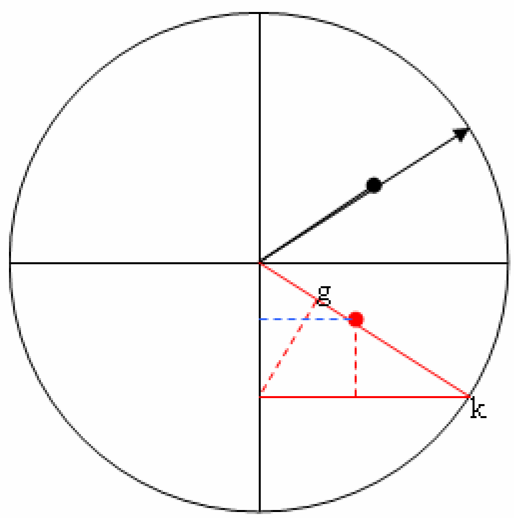
Comparing Delta Thetas

Chapter 30 presented the known property for small angles which states that when we measure an angle in radians , the angle equals its own tangent. We need that relationship in this chapter because we can demonstrate a proportion between the tangents of small angles swept by planets in their orbits. But we want to get at the proportion between the angles themselves - not the proportion between their tangents. So the previous chapter solved that dilemma by demonstrating that the angle and its tangent are one and the same. Of course this is only true for small angles. But that is perfectly fine for our purposes because we are going to look at extremely tiny angles at the very moment that our various planets are at position b at the end of the semiminor axes of the elliptical orbits.

We are going to return to the hodograph velocity diagram on which we represented the orbits of two planets whose orbital semimajor axes were the same but whose orbital eccentricity differed. We found that this hodograph told us to expect that $V_t \propto b^2$. I hinted that we would find this to be impossible and in this chapter we will begin to see why.

Let's examine instants of time when the planet is at position *b* at the top of the semiminor axis.
Compare the planet in a circular orbit to a planet in an elliptical orbit by inspecting their representations on the same hodograph.



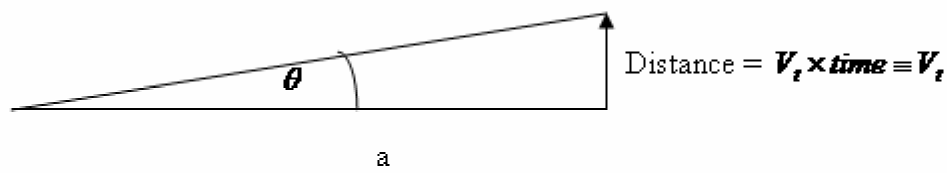
Note that the circular orbit is represented in black. The planet is at the midpoint of its velocity arrow as would be dictated by the Inverse Proprtion Machine. The total velocity is all tangential so the total velocity is equal to the tangential velocity for the circular orbit. Recall that we named the velocity that equals the velocity of the radius of the velocity circle V_2 . This , by inspection is equal to the velocity of the circular orbit. Also, it is evident that the distance to the Sun never changes for the circular orbit and thus the planet is always at position b for the circular orbit.

Notice that the planet at postion b in the elliptical orbit is represented in red. The length of the semiminor axis , b , is represented by the dashed blue line. The tangential velocity is represented by the segment $\overline{gk}$ and is noticeably smaller than the tangential velocity for the circle that is represented by the entire length of the solid black arrow. In fact in a previous chapter we found that the hodograph misleads us to believe that $V_t \propto b^2$.

Let's portray the wedges that are swept by the two planets so we can compare the angles each planet sweeps in a tiny unit of time.

The wedge of area swept by a planet has a base and a far side. We are only considering the planet when it is at position b so the base will be equal for both planets. That distance will be equal to a,(since the planet is always at a distance ,a, from the Sun when it is at b) the semimajor axis of the ellipse which equals the radius of the circular orbit. (Recall that our hodograph is for orbits of equal semimajor axes and that the circular member of this family of orbits has a radius equal to that semimajor axis length.)

So the base of the wedge is the same. The far side is determined by the distance the planet travels in the tangential direction at a right angle to the base. We are considering a small unit of time. So the distance traveled is determined by the tangential velocity multiplied by that unit of time. For example if the planet's tangential velocity is 100 kilometers per second and our time unit is two seconds, our planet travels 200 kilometers in a tangential direction.



To make the math easy, assign our tiny time interval to be equal to 1 second . Then our distance on the far side of the wedge can be represented by the tangential velocity itself if the velocity is expressed in units of kilometers per second.

The hodograph told us that $V_t \propto b^2$. So if the far side of the wedge for the circular orbit is V_2, the tangential velocity for the circular orbit, then for the elliptical orbit - with smaller b than the circular orbit:

The far side of the wedge for the elliptical orbit will be $V_2 \times \frac{b^2{}_{ellipse}}{b^2{}_{circle}}$.

Now the tangent of the angle θ of the wedge is equal to the far side divided by the base.

For the circular orbit $\tan\theta = \dfrac{V_2}{a}$

For the elliptical orbit $\tan\theta = \left(\dfrac{V_2}{a}\right)\dfrac{b^2{}_{ellipse}}{b^2{}_{circle}}$

We want to get the ratio of the angles.

$$\frac{\tan\theta_{circle}}{\tan\theta_{ellipse}} = \frac{V_2}{a} \div \left(\frac{V_2}{a}\right)\left(\frac{b^2{}_{ellipse}}{b^2{}_{circle}}\right) = \frac{b^2{}_{circle}}{b^2{}_{ellipse}}$$

And so we have a ratio for the tangents of the angles swept by the two orbiting planets at b.

In Chapter 30 we showed that for small angles the angle is equal to its own tangent if we measure the angle in radians. Our angle is indeed tiny since we look at only the instants in time when the planet is at position b. We can measure our angle swept any way we want to so if we decide to measure it in radians, that is fine also. Thus the ratio of the tangents of our angles is the same ratio for the angles themselves. So we conclude that:

For the planets at b, in a tiny time period:

$$\theta_{circle} = \theta_{ellipse}\left(\frac{b^2{}_{circle}}{b^2{}_{ellipse}}\right)$$

And this marvelous conclusion is wrong. It will reveal to us how to scale the hodograph. The conclusion will be proven to be incorrect by demonstrating what it would say about the effect of force on the planet at position b. It would give us false and impossible results. We will show all this in the next chapter.