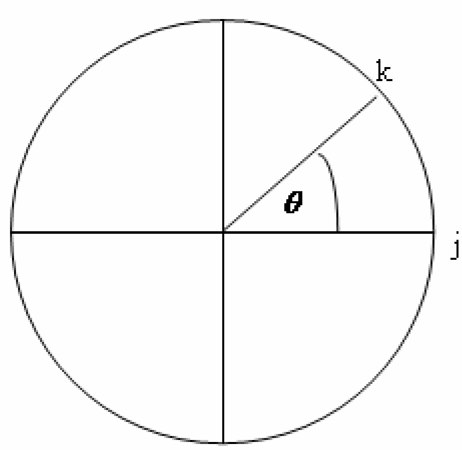
Tangents of Small Angles

In this chapter we will demonstrate a known property of tangents. The tangent of a small angle is equal to the angle itself if the angle is measured in radians. In a Chapter 31 we will use this property when we examine the small angle swept in the instant that the planet is at position b.

First let's explain how to measure angles in radians. We are accustomed to designating an angle for a wedge of a circle by simply stating the number of degrees that the wedge spans at the center of the circle.



But we could also describe the same angle by the length of the arc it cuts along the circle which is $arc\,\overline{jk}$.

Now the whole distance around the circumference of the circle is $2\pi R$ and we can define the radius of our circle to be equal to 1. In that case the circle is referred to as a unit circle. The angle that would be spanned if we were to travel once around the circle is 360 degrees. In radian measure the arc cut would be the distance along the circumference of the circle. We could measure it with a ruler if we could straighten it out and from our formula for circumference we know the distance would measure 2π times the radius of the circle. And so the convention is to say that is 2π radians long. So the angle expressed in radians that spans one trip around a circle is 2π . Now if we only travel a smaller angle, 50 degrees for example, then we have travelled $\frac{50}{360}$ times less than one trip around and we have travelled the same fraction of the total arc of circumference. The angle expressed in radians would be $\left(\frac{50}{360}\right)2\pi$.

Note the subtle linguistic distinction. There are two ways to state the same notion:

"The angle expressed in radians is $\left(\frac{50}{360}\right)2\pi$."

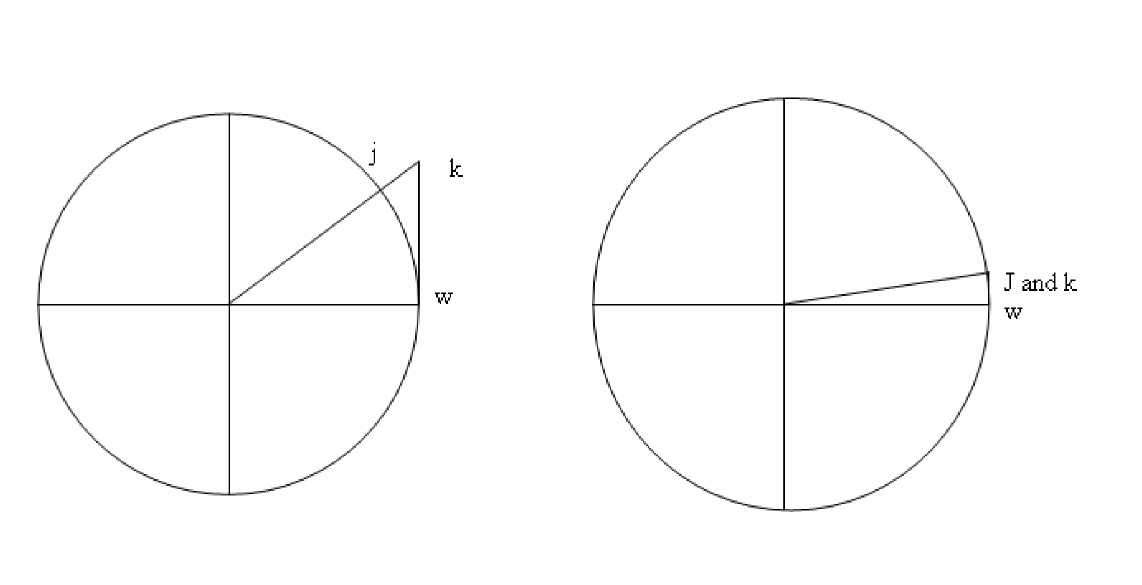
"The angle is $\left(\frac{50}{360}\right)2\pi$ radians."

Furthermore, the linguistic subtlety affects the second statement when we know that we are dealing with a unit circle whose radius is equal to one. In that case as long as we know we are speaking of the angle as a fraction of the total circumference, or in other words in terms of radians, we can also state:

"The angle is $\left(\frac{50}{360}\right)2\pi$."

Now examine the unit circle for which the radius is equal to one. Notice that the tangent of the central angle of the wedge is equal to the length of the segment $\overline{wk}$ since tangent is defined as the opposite side divided by the adjacent side and the adjacent side- being the radius of the unit circle - is equal to 1.

Notice that as the central angle gets smaller, the length of the arc $\overline{wj}$ approaches the length of the segment $\overline{wk}$. For a tiny central angle they are essentially equal.



So as the central angle becomes tiny, the segment $\overline{wk}$ (which we know is the tangent of the central angle in our unit circle) becomes essentially equal to arc $\overline{wj}$ spanned by the central angle.

Now if we express the tiny angle in radian form , we are in essence stating its length along the circumference of the circle and that length is $\overline{wj}$. But when the central angle is tiny, $\overline{wj} \approx \overline{wk}$. In other words the angle expressed in radians, as $\overline{wj}$, is equal to the tangent, $\overline{wk}$, of the angle itself.

And this is what we set out to demonstrate. - When the angle is expressed in radians and the angle is small, the tangent of an angle is equal to the angle itself:

For a small angle;

$$\tan\theta = \theta$$