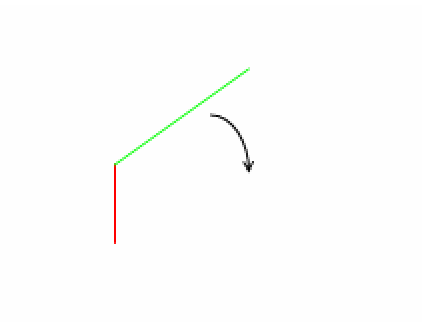
The Hodograph at b

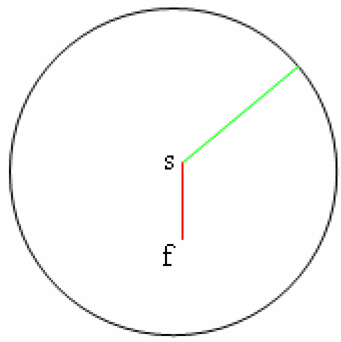
In this chapter we will examine the hodograph for orbits for planets within a solar system. We look at the special case for which the orbits all have the same length semimajor axis but different length semiminor axes. We study the planets in these orbits when they are at position b, at the end of the semiminor axis. We draw the hodograph for these different orbits on the same velocity diagram. We will see that there are, in the hodograph, apparent relationships between total velocity, tangential velocity, and the length of the semiminor axis. We will use these apparent relationships to find the correct method for scaling the hodograph for elliptical orbits. We learned in Chapter 26 that for circular orbits, scaling is necessary. Now we will begin to do so for elliptical orbits. We will do so by revealing a falsehood concerning the apparent proportion between tangential velocity and the length of the semilatus rectum. In other words in this chapter we will show that the hodograph tells us that tangential velocity is directly proportional to the length of the semilatus rectum for orbits of equal semimajor axis length. But in the next chapter we will show how this can not be the case. And subsequently in the next several chapters we will show how this inaccuracy indicates how to scale the hodograph for elliptical orbits correctly.

Let's review the basics of the hodograph that we will need for this chapter.

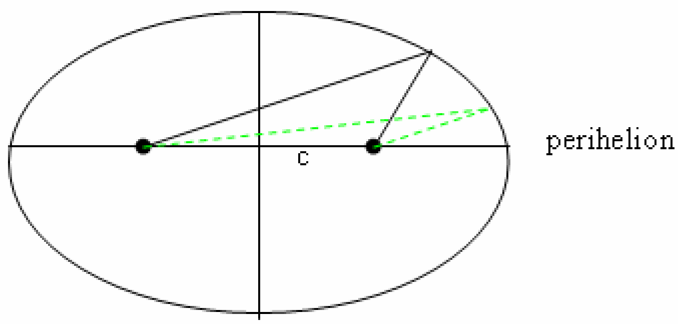
The long segment of the inverse proportion machine rotates around the short segment to produce the velocity circle of the hodograph.



The hodograph velocity circle thus has a radius equal to the length of the long segment. The length of the short segment determines the position of the second focus, f, in the figure below. The Sun is at the other focus s, in the center of the velocity circle below:



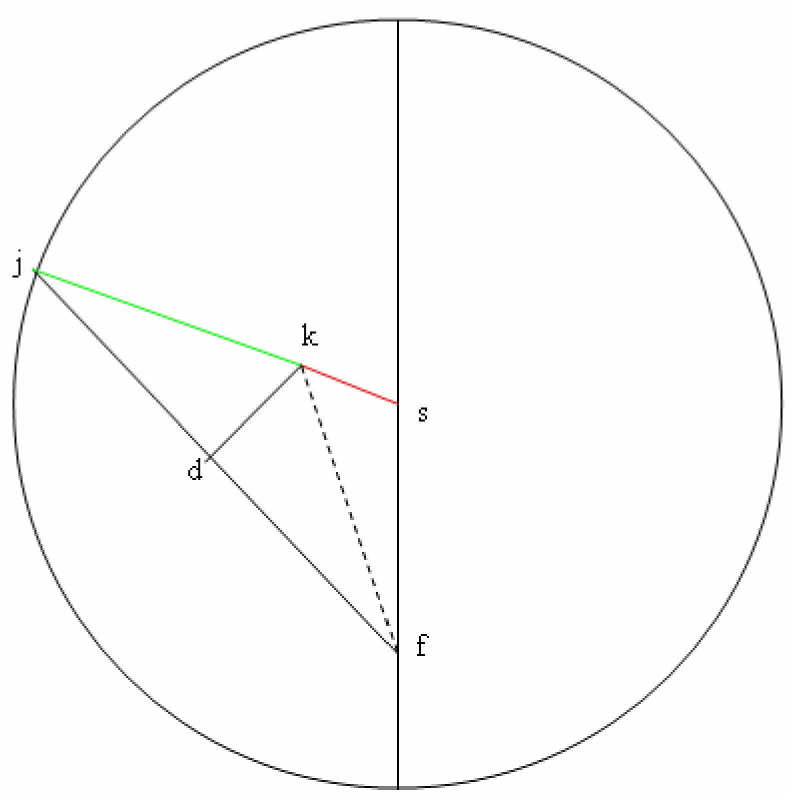
Now look at the figure below which represents the string and tack method of creating an ellipse. We drag the string around clockwise to create the ellipse - for example the string is represented in black and then in green. Moments later ,the string would be in the horizontal position corresponding to the perihelion position. It is difficult to portray the string in this position because the string segment from the distant focus overlaps the string segment from the closer focus as both segments meet at perihelion position. But the point is that in that position it is evident that the total length of the string is equal to $2a$. The segment from the long segment measures $a+c$ and the short segment's length is $a-c$ so their sum is $(a+c)+(a-c)=2a$.



Now that we see the ellipse in the string and tack context, we will look at the string and tack in the hodograph and demonstrate that the length of the string is equal to the radius of the velocity circle.

In the figure below $\overline{fs}$ is the short segment and $\overline{sj}$ is the long segment of the Inverse Proportion Machine.

The planet is at position k . The Sun is at the first focus of the ellipse at position, s. The point s is also the center of the velocity circle. The Inverse Proportion Machine sets point d at the midpoint of segment $\overline{fj}$ and sets the angle , $\angle fdk$. To be a right angle. Two equal right triangles result since they share side $\overline{dk}$ and since segment $\overline{fd}$ is equal to segment $\overline{dj}$ and since the angle between these two equal sets of sides is a right angle in each case. These equal triangles are Δfdk and Δjdk . Since these triangles are equal, their hypoteni are equal, so segment $\overline{fk}$ is equal to segment $\overline{jk}$.



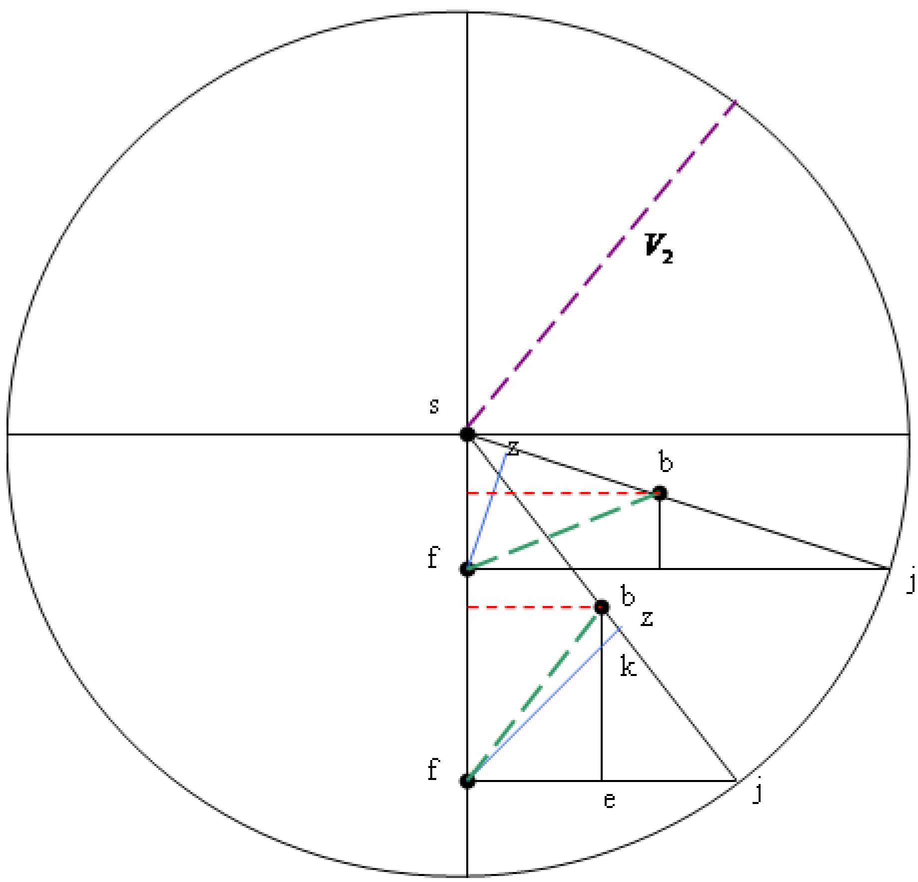
In a previous chapter we showed that the Inverse Proportion Machine contains the string of the string and tack ellipse. The total length of the string in the hodograph above is the path $\overline{fks}$.

Note that segment $\overline{sj}$ is the radius of the velocity circle. Now since segments $\overline{sk}+\overline{kj}=\overline{sj}$ and since segment $\overline{fk}=\overline{jk}$, then the sum of the segments $\overline{fk}+\overline{sk}$ is also equal to the radius of the velocity circle. These segments are the two segments of the string of the string and tack ellipse. So since the two segments of the string sum to $2a$, the radius of the velocity circle is also equal to $2a$. This holds true regardless of how eccentric the orbit is as long as the orbit is drawn on the same hodograph diagram because the triangles, segments, and angles always behave as described in the preceding paragraphs. The length of the string will always equal $2a$ which will always be the length of the radius of the velocity circle.

Now we can return to the examination of the hodograph in the context of the planet when it is at position b.

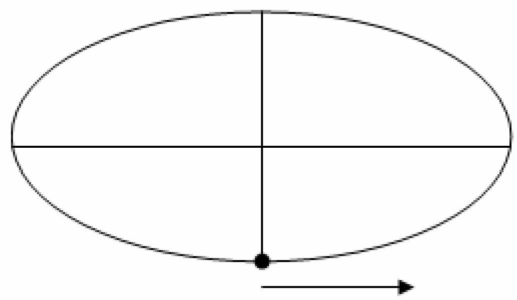
We showed in a previous chapter that when the planet is at either end of the semiminor axis at position b, the distance to the Sun is equal to a, the semimajor axis.

In the hodograph velocity circle below we have represented the case for the planet at position b for two orbits. As shown above, since both orbits are on the same velocity diagram their semimajor axes will be of equal length. The purple dashed radius of the circle will be explained below.



The two orbits share one focus, s. The second focus is denoted by f for each orbit. The segments $\overline{fj}$ represent the total velocity for each orbit when its planet is at position b. Note that for the more eccentric orbit, the segments $\overline{fz}$ and $\overline{zb}$ have crossed compared to their orientation to each other in the less eccentric orbit. This is important because it tells us that we will have to treat eccentric orbits separately in our mathematical discussion below. The triangles that we analyze for the less eccentric orbit are not present in the proper orientation in the more eccentric orbit. But we will get to that later in this chapter.

The Inverse Proportion Machine dictates the arrangement of the hodograph diagram above for the planet at position b but we can intuitively surmise how to create the arrangement. We know that for all velocity hodographs the total velocity vector has its origin at the second focus. For any elliptical orbit when the planet is at either end of the semiminor axis the planet by inspection must be traveling parallel to the major axis of the ellipse:



So when the planet is at position b we draw the total velocity arrow, $\overline{fj}$, horizontally with its origin at the second focus. Note also that for both orbits the semiminor axis is represented by the red dotted line drawn to the planet when it is at position b.

Recall that in Chapter 11 we showed that the Inverse Proportion Machine generates the segment that represents the tangential velocity - which is the segment $\overline{zj}$ in the hodograph above. Similarly, the Inverse Proportion Machine generates the radial velocity represented by the segment $\overline{zf}$.

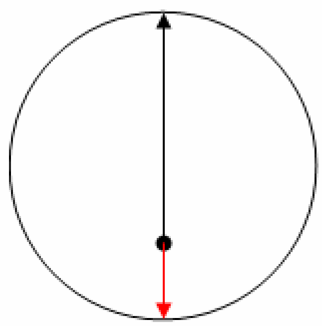
Now that we have reviewed the relevant aspects of the hodograph and described the arrangement for the planet at position b we can proceed to make our observations. We will first use our Inverse Proportion Right Triangle to demonstrate some relationships that are present in the less eccentric orbit.

But first let's pause to recall that in Chapter 27 we introduced some arbitrary naming conventions. Some of the parts of the hodograph were assigned letter labels:

V_t is the tangential velocity.

V_3 is the total velocity

V_2 , represented by the purple dashed line in the hodograph above, is the radius of the velocity hodograph. It is also half of the sum of the total velocity at perihelion added to the total velocity at aphelion. This is easy to see since their total velocities represented by V_3 sum to be the diameter of the velocity circle:



In the velocity diagram above the total velocity at perihelion is in black and the total velocity at aphelion is in red. Recall that the Inverse Proportion Machine generates the total velocity arrow which reaches from the second focus outward to the velocity circle. The perihelion and aphelion arrangements are each generated by positioning the Inverse Proportion Machine's two segments at zero and 180 degrees from each other respectively. If this is unclear, these positions can be reviewed by reading the relevant parts of a previous chapter.

Now, back to the hodograph for the two orbits whose planets are at position b. Note that the Small Inverse Proportion Triangle is represented by the triangle Δzfj for the less eccentric orbit.

By inspection of the red dashed line and the segment $\overline{fj}$ for the less eccentric orbit we see that since $\overline{fj}$ is bisected via the action of the Inverse Proportion Machine and since $\overline{fj}$ is V_3:

$b \propto V_3$ since $b = \frac{1}{2}V_3$ by inspection of the hodograph.

Now apply the rule of our Small Inverse Proportion Triangle from our previous chapter that states " For any right triangle whose hypotenuse is bisected perpendicularly, the smaller hypotenuse times the original base that contains it, times two, is equal to the original hypotenuse squared. " This tells us that in the hodograph velocity circle above for the planet in an orbit of low eccentricity at b:

$$\overline{fj}^2 = 2 \times \overline{zj} \times \overline{bj}$$

The difficulty now is that the more eccentric orbit does not contain the Small Inverse Proportion Triangle so we need to demonstrate that the above equation also holds for eccentric orbits as well. We will accomplish this using sines and cosines of angles. Once we have done so we can generalize and say that the above formula is true at position b for all orbits.

For the eccentric orbit in the hodograph above:

$\angle kfe = \angle zfj$ since these represent the angle shared by triangles Δfke and Δfzj.

Now it will be shown that $\angle ekf = \angle ejb$:

Note that Δfke is similar to Δbkz since they are right triangles and $\angle fke = \angle bkz$.

Note that Δbkz is similar to Δbej since $\angle kbz$ is shared as $\angle ebj$.

So Δfke is similar to Δbej.

In any right triangle the sum of the angles is 180 degrees and one angle is a right angle so the other two angles add up to 90 degrees.

$\angle kbz = 90 - \angle bkz$ by inspection of triangle Δzbk.

$\angle ejb = 90 - \angle kbz$

So by substituting for $\angle kbz$:

$\angle ejb = 90 - (90 - \angle bkz) = \angle bkz$

So $\angle ejb = \angle bkz$

Since as we showed above, $\angle bkz = \angle fke$:

$\angle ejb = \angle fke$

So far we have two sets of equal angles:

$\angle ejb = \angle fke$ and $\angle kfe = \angle zfj$.

Sines of equal angles are equal. Cosines of equal angles are equal.

$\sin\angle kfe = \sin\angle zfj$ so:

$$\frac{\overline{ke}}{\overline{fk}} = \frac{\overline{zj}}{\overline{fj}}$$

$\cos\angle ejb = \cos\angle fke$ so:

$$\frac{\overline{ej}}{\overline{bj}} = \frac{\overline{ek}}{\overline{fk}}$$

So by substitution:

$$\frac{\overline{ek}}{\overline{fk}} = \frac{\overline{ej}}{\overline{bj}} = \frac{\overline{zj}}{\overline{fj}}$$

So by multiplying out:

$$\overline{ej} \times \overline{fj} = \overline{bj} \times \overline{zj}$$

Now, in general $x\left(\frac{x}{2}\right)=\frac{x^2}{2}$. In other words any quantity times half of itself is equal to itself squared and then divided by 2. And note that $\overline{ej}=\frac{\overline{fj}}{2}$. So:

$$\overline{ej}\times\overline{fj}=\frac{\overline{fj}^2}{2}$$

Thus $\frac{\overline{fj}^2}{2}=\overline{bj}\times\overline{zj}$ or as we set out to prove:

$$\overline{fj}^2=2\overline{bj}\times\overline{zj}$$

And so we have shown the above equation is valid for orbits of both small and large eccentricities when the planet is at position b. Let's continue to study the planet when it is at position b. Let's apply the above equation to the types of velocity that its terms represent.

Since the distance to the Sun is equal to a when the planet is at position b, and by the relations we showed above for the string and tack ellipse:

$a = \overline{sb} = \overline{fb} = \overline{bj}$ so we can substitute a for $\overline{bj}$.

We showed above that V_t is represented by $\overline{zj}$ and V_3 is represented by $\overline{fj}$ so:

$\overline{fj}^2 = 2\overline{bj} \times \overline{zj}$ becomes:

$V_3^2 = 2 \times a \times V_t$.

Rearranging:

$$V_t = \frac{V_3^2}{2a}$$

But we found above by mere inspection of the hodograph that $b \propto V_3$ so if we change to from equations to proportions:

$$V_t = \frac{V_3^2}{2a} \propto \frac{V_3^2}{2a} \propto \frac{b^2}{2a} \propto \frac{b^2}{a} \quad .$$

Note that we dropped the 2 in the denominator of our proportion since 2 is a constant.

Recall that in Chapter 2 we showed that for the semilatus rectum , p , of an ellipse:

$$p = \frac{b^2}{a}$$

So our proportion above becomes:

$V_t \propto p$ **when the planet is at** b **.**

Be aware that the above proportion only applies for the hodograph that contains orbits of equal semimajor axis when the planets are at position b .

This proportion is what we wanted to produce. It is the proportion that our hodograph tells us to expect. But it is false. And in the next chapter we will compare what is false here to what is true. And that will tell us how to scale our hodograph for elliptical orbits.