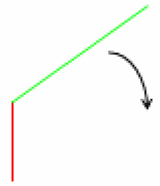
Total and Tangential Velocity

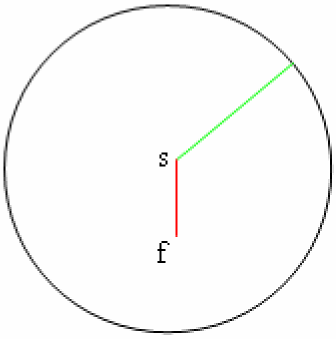
In this chapter we will examine the hodograph for planets within a solar system to find a relationship between total velocity and tangential velocity. . We will use this valid information to find the correct method for scaling the hodograph for elliptical orbits. We will also use this relationship in later chapters in *a priori* proofs of Kepler's Third Law and of the Energy Equation for elliptical orbits.

Let's review the basics of the hodograph that we will need for this chapter.

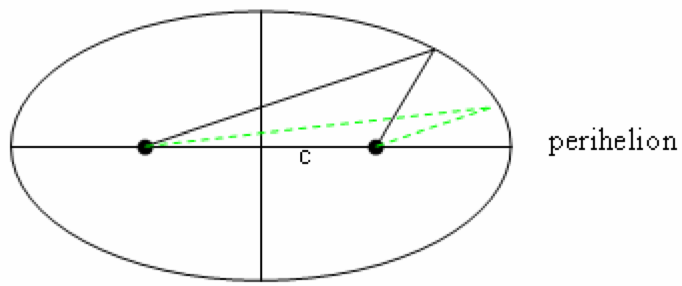
The long segment of the inverse proportion machine rotates around the short segment to produce the velocity circle of the hodograph.



The hodograph velocity circle thus has a radius equal to the length of the long segment. The length of the short segment determines the position of the second focus, f, in the figure below. The Sun is at the other focus s, in the center of the velocity circle below:



Now look at the figure below which represents the string and tack method of creating an ellipse. We drag the string around clockwise to create the ellipse - for example the string is represented in black and then in green. Moments later ,the string would be in the horizontal position corresponding to the perihelion position. It is difficult to portray the string in this position because the string segment from the distant focus overlaps the string segment from the closer focus as both segments meet at perihelion position. But the point is that in that position it is evident that the total length of the string is equal to $2a$. The segment from the long segment measures $a+c$ and the short segment's length is $a-c$ so their sum is $(a+c)+(a-c)=2a$.

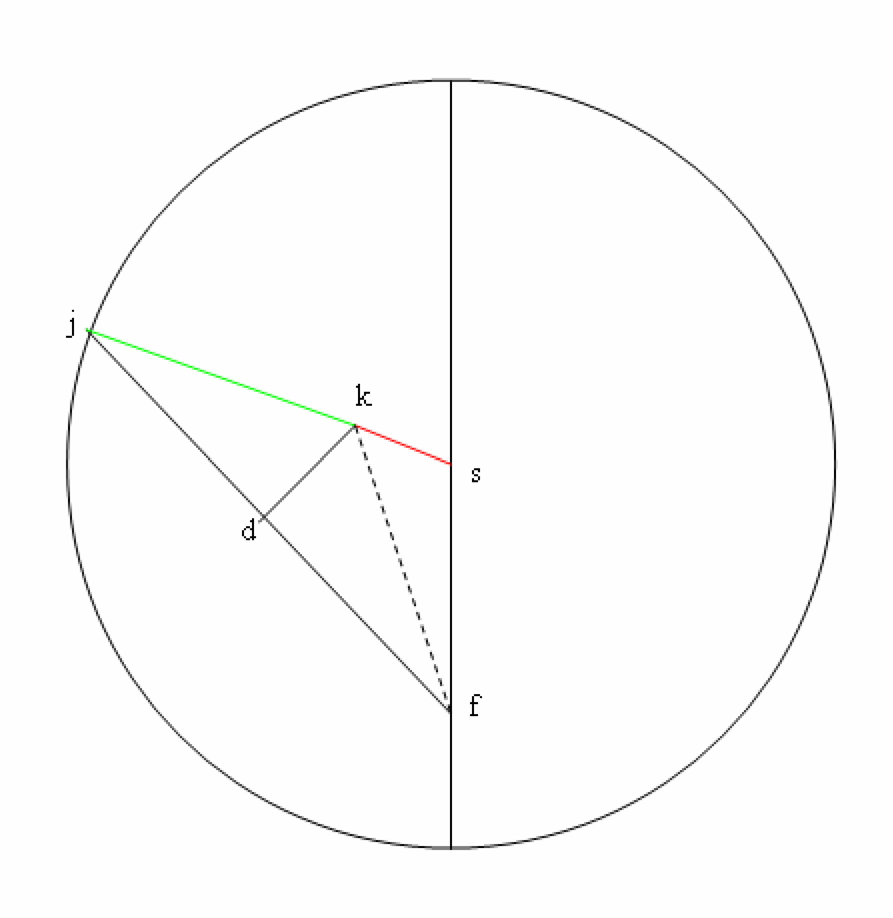


Now that we see the ellipse in the string and tack context, we will look at the string and tack in the hodograph and demonstrate that the length of the string is equal to the radius of the velocity circle.

In the figure below $\overline{fs}$ is the short segment and $\overline{sj}$ is the long segment of the Inverse Proportion Machine.

The planet is at position k . The Sun is at the first focus of the ellipse at position, s. The point s is also the center of the velocity circle. The Inverse Proportion Machine sets point k st the midpoint of segment $\overline{fj}$ and sets the angle , $\angle fdk$. To be a right angle. Two equal right triangles result since they share side $\overline{dk}$ and since segment $\overline{fd}$ is equal to segment $\overline{dj}$ and since the angle between these two equal sets of sides is a right angle in each case.

These equal triangles are Δfdk and Δjdk. Since these triangles are equal, their hypoteni are equal, so segment $\overline{fk}$ is equal to segment $\overline{jk}$.

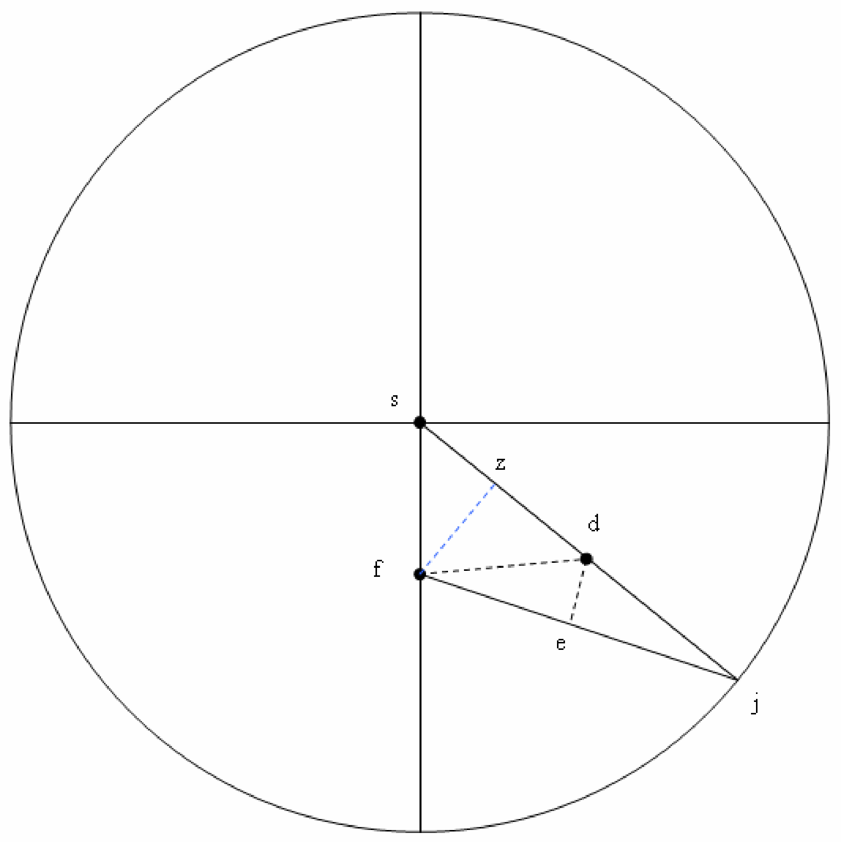


In Chapter 11 we showed that the Inverse Proportion Machine contains the string of the string and tack ellipse. The total length of the string in the hodograph above is the path $\overline{fks}$.

Note that segment $\overline{sj}$ is the radius of the velocity circle.

Now since segments $\overline{sk}+\overline{kj}=\overline{sj}$ and since segment $\overline{fk}=\overline{jk}$, then the sum of the segments $\overline{fk}+\overline{sk}$ is also equal to the radius of the velocity circle. These segments are the two segments of the string of the string and tack ellipse. So since the two segments of the string sum to $2a$, the radius of the velocity circle is also equal to $2a$. This holds true regardless of how eccentric the orbit is as long as the orbit is drawn on the same hodograph diagram because the triangles, segments, and angles always behave as described in the preceding paragraphs. The length of the string will always equal $2a$ which will always be the length of the radius of the velocity circle.

As shown above, since both orbits are on the same velocity diagram their semimajor axes will be of equal length.



Hodograph A

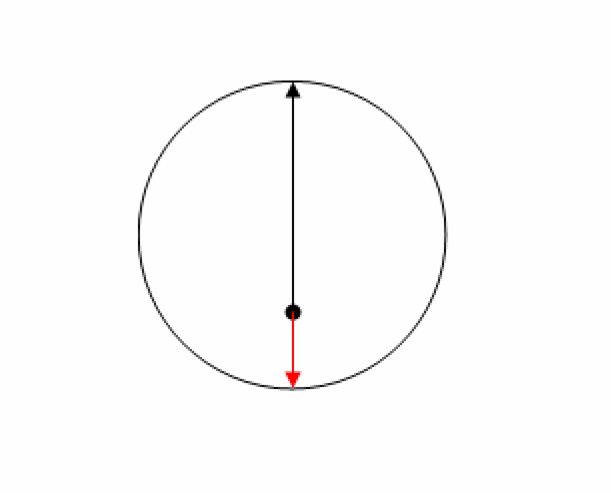
Recall that in Chapter 11 we showed that the Inverse Proportion Machine generates the segment that represents the tangential velocity - which is the segment $\overline{zj}$ in the hodograph above. Similarly, the Inverse Proportion Machine generates the radial velocity represented by the segment $\overline{zf}$.

Let's pause to introduce some arbitrary naming conventions. I found it easier to label some of the parts of the hodograph with letter labels. I will use these names in the text:

V_t is the tangential velocity.

V_3 is the total velocity

V_2 is the radius of the velocity hodograph. It is also half of the sum of the total velocity at perihelion added to the total velocity at aphelion. This is easy to see since the total velocities represented by V_3 sum to be the diameter of the velocity circle:



In the velocity diagram above the total velocity at perihelion is in black and the total velocity at aphelion is in red. Recall that the Inverse Proportion Machine generates the total velocity arrow and that the arrow reaches from the second focus outward to the velocity circle. The perihelion and aphelion arrangement is generated by positioning the Inverse Proportion Machine's segments at zero and 180 degrees from each other respectively.

Note that the Inverse Proportion Machine's Right Triangle is represented by the triangle Δzfj. Note that segment $\overline{ed}$ bisects segment $\overline{fj}$ as mandated by the Inverse Proportion Machine. So we can apply the rule of our Small Inverse Proportion Triangle from our previous chapter that states " For any right triangle whose hypotenuse is bisected perpendicularly, the smaller hypotenuse times the original base that contains it, times two, is equal to the original hypotenuse squared. "

This tells us that in the elliptical orbit in the first hodograph labeled "hodograph A" :

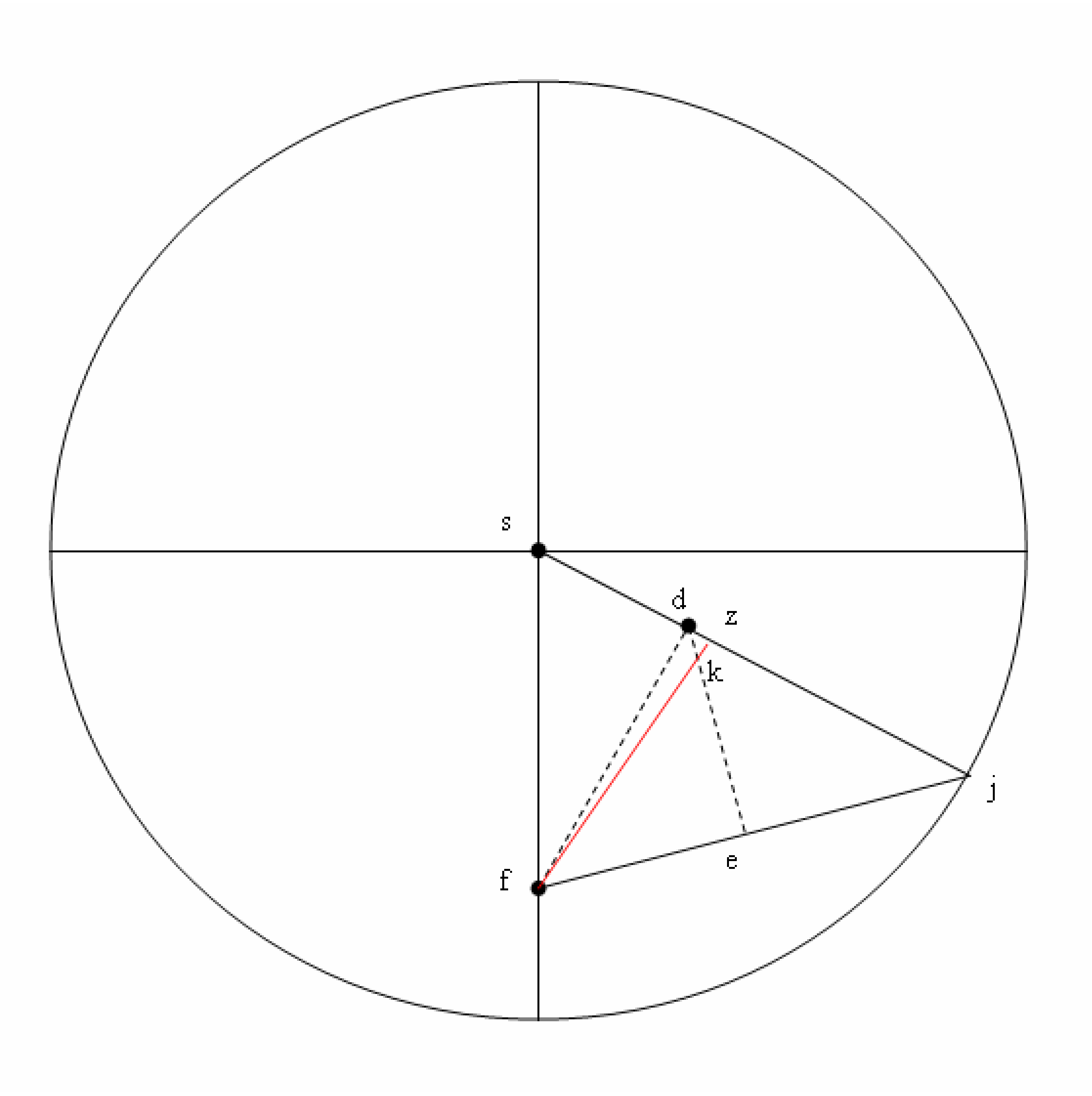
$$\overline{fj}^2 = 2 \times \overline{zj} \times \overline{dj}$$

From the above equation we learn that the hodograph tells us that since $\overline{fj}$ represents total velocity which we refer to as V_3, and since $\overline{zj}$ represents tangential velocity which we call V_t, and since $\overline{dj}$ by inspection is the distance from the second focus to the planet which must equal $(2a-r)$ where r is the distance from the Sun to the planet:

$$V^2{}_3 = 2V_t(2a-r)$$

The above relationship will be useful in later chapters when we will use it to show how to properly scale hodographs to each other and in deriving the Energy Equation for elliptical orbits.

But we must first back up to show the geometry for the above derivation when the orbit is highly eccentric. The problem with the derivation above is that for eccentric orbits, the right triangle that we subject to our Small Inverse Proportion Triangle analysis disappears. This is not a problem since we can derive the above same equation for the eccentric orbit by using sines, cosines, and tangents of angles instead of using the Small Inverse Proportion Triangle analysis technique.



So note in the hodograph above that there are right angles $\angle def$ and $\angle dej$ and $\angle fzj$ and $\angle fzd$.

Note that angle $\angle kfe = \angle zfj$ and that each angle is in a right triangle. The two angles are equal and by the rules of geometry their sines must be equal so;

$$\sin\angle kfe = \sin\angle zfj = \frac{\overline{ek}}{\overline{fk}} = \frac{\overline{zj}}{\overline{fj}}$$

Note that triangles Δfjz and Δfke are right triangles that share the angle $\angle kfe$. Thus their third angle must be equal so that $\angle fke = \angle zjf$. Since these angles are equal their cosines must be equal so that:

$$\cos\angle fke = \cos\angle zjf = \frac{\overline{ek}}{\overline{fk}} = \frac{\overline{ej}}{\overline{dj}}$$

So note that we have found the equalities:

$$\frac{\overline{ek}}{\overline{fk}} = \frac{\overline{zj}}{\overline{fj}} = \frac{\overline{ej}}{\overline{dj}}$$

Rearranging the last two terms in the above equation:

$$\overline{zj} \times \overline{dj} = \overline{fj} \times \overline{ej}$$

Note that by the inverse proportion machine that $\overline{ej} = \frac{\overline{fj}}{2}$.

We can show by simple algebra that in general a quantity times half of itself is equal to the original quantity squared, divided by two:

$$x\left(\frac{x}{2}\right) = \frac{x^2}{2}$$

So $\overline{fj} \times \overline{ej} = \frac{\left(\overline{fj}\right)^2}{2}$

And so we have shown that $\overline{zj} \times \overline{dj} = \frac{\left(\overline{fj}\right)^2}{2}$.

And this is what we set out to prove. By inspection of the way the Inverse Proportion Machine has set up the hodograph above, we see that $\overline{zj}$ represents V_t, the tangential velocity. We see also that $\overline{dj}$ since it is equal to the segment $\overline{fd}$, represents the distance from the second focus to the planet which we know from the above discussion to be equal to

$2a - r$. And lastly we see also that $\overline{fj}$ represents total velocity which we call V_3 by convention in this book.

And so we have shown for eccentric ellipses also, that :

$$\frac{V^2{}_3}{2} = V_t(2a - r)$$

Reaaranging to match the form of the formula exactly as expressed above that we derived for orbits that are less elliptical than our present one:

$$V^2{}_3 = 2V_t(2a - r)$$

In the next chapters we will put this relationship to good use.