Scale the Circular Hodograph

It is easy to understand why there is a need to scale one diagram to another when we compare two velocity diagrams. Let's consider the case of planets in circular orbits in a solar system. We have shown that there is a proportion between the period of an orbit and the radius of an orbit for the case where the orbits are circular. In the paragraphs below it will be shown that the velocity of the planet decreases inversely proportional to the square root of the radius of the orbit - or more simply put, that the velocity is less for planets that are farther from the Sun. However, see what happens when we draw the hodograph two planets, one close to the Sun and one far away. The planet that is farther away will have a larger hodograph circle since the radius is bigger. It's velocity vectors will therefore be larger than those for the smaller orbit. We see immediately that this is improper since the velocity is actually smaller for the planet that is farther from the Sun. Hence there is a need to scale the hodographs when we compare them.

In this chapter I will introduce a novel procedure, "scaling the hodograph", which I call "scalometry" that serves greatly to explain the orbital behavior of planets.

In particular the scaling technique allows us to compare different size orbits around the same central body. It will reveal the correct method of interpreting the velocity of planets that share a solar system. This is critically important to our *a priori* scheme to explain orbits. We can already show *a priori* the relationship between the size and the period for circular orbits.

Scalometry will have tremendous importance. Once we have a correct method for scaling the hodograph we will apply it in later chapters to the more general hodograph of the elliptically shaped orbit. And then ultimately it will lead to the *a priori* derivation of the Kepler's Third Law for elliptical orbits. Subsequently, all will fall into place to explain the Energy Equation as well.

As you read subsequent chapters, and as the explanation of orbits becomes clearer, bear in mind that the general scheme is accomplished using the Inverse Proportion Machine and the hodograph scaling technique.

We know from a previous chapter that $T \propto R^{\frac{3}{2}}$ for circular orbits, Kepler's Third Law.

We also know that the length of the path for circular orbits is $2\pi R$.

Hence the velocity, which is constant for a circular orbit is $\frac{2\pi R}{T}$.

But using Kepler's Third Law which we have shown in *a priori* fashion and substituting the proportion $R^{\frac{3}{2}}$ for T in the denominator above:

The velocity for circular orbits within a solar system obey the proportion:

$v \propto \frac{2\pi R}{R^{\frac{3}{2}}} \propto \frac{2\pi R}{\sqrt{R}\sqrt{R}\sqrt{R}} \propto \frac{2\pi}{\sqrt{R}}$ and since 2π is a constant:

$$v \propto \frac{1}{\sqrt{R}}$$

So we see that for planets in the same solar system the velocity decreases as the radius of the circular orbit increases. Planets that are further from the Sun travel more slowly. More exactly the velocity decreases in proportion to the square root of the radius. For example, a planet that is 25 times further from the Sun will travel 5 times more slowly compared to the closer planet.

We can digress for a moment into empiricism to note that in Chapter 22 we used Gallileo's empirical finding that greater masses fall at the same rate as lesser masses. This led us to fill in the regulator of the proportion

$F \propto \frac{1}{R^2}$ resulting in the full equation $F = \frac{GMm}{R^2}$. If we use the same empirical findings to describe the period as we did in a previous chapter where we showed, $T = \frac{2\pi}{\sqrt{GM}} R^{\frac{3}{2}}$

Then we can substitute the left side of the equation above in the denominator, instead of substituting merely the proportion for T, and we obtain the more complete description of velocity in a circular orbit:

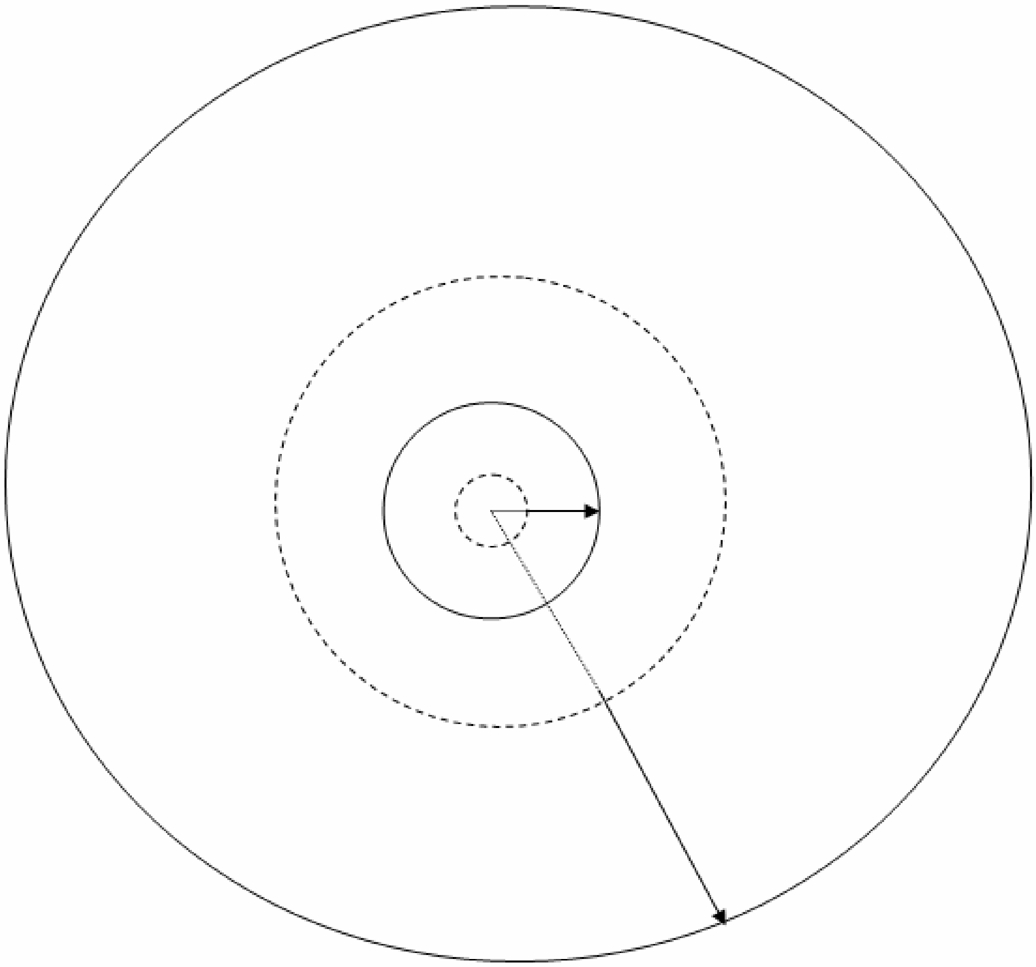
$v = \frac{2\pi R}{\frac{2\pi}{\sqrt{GM}} R^{\frac{3}{2}}} = \frac{2\pi R\sqrt{GM}}{2\pi\sqrt{R}\sqrt{R}\sqrt{R}} = \sqrt{\frac{GM}{R}}$ and this is in fact in agreement with the physics texts.

But returning to our scaling method we found *a priori* above that for orbits in the same solar system:

$$v \propto \frac{1}{\sqrt{R}}$$

Let's see what happens when we try to draw the hodograph for two different size orbits in the same solar system on the same hodograph diagram. We will draw, for an example, the hodograph of two planets on the same diagram for the situation wherein one planet is four times farther away from the Sun than the other planet. In the

diagram below the radius of each planet is represented by the dashed segment drawn to meet the orbit which in turn is represented by a dashed circle. The velocity arrow is slightly more complicated to represent since it overlaps the dashed radius. (Recall from Chapter 13 that we like to represent the velocity and the radius on the same straight line in compliance with the Inverse Proportion Machine.) But in any case, let the velocity arrow be represented as starting at the center of the circle by the dashed segment and additionally by the solid arrow extended from the dashed segment, so that the velocity arrow actually reaches all the way from the center of the diagram out to touch the appropriate
solid circle.



We see that as the radius of the orbit increases the velocity arrow increases in a one to one proportion. In other words if we took what we see literally to be correct, as the radius of the planet increases fourfold the velocity also quadruples. But we just showed this is not the case. The velocity needs to decrease in proportion to the square

root of the radius so as the radius increases fourfold the velocity is twice as small for the planet that is four times farther away. So here is our great clue as how to scale for circular orbits. We must scale what we see literally. The result of scaling must agree with $v \propto \frac{1}{\sqrt{R}}$.
This is accomplished by inspection of the hodographs in two step fashion.

In the first step, scale down the velocity arrow that is too large by equalizing it to the smaller arrow by multiplying the long arrow by $\frac{R}{R^{'}}$. The second step is to make it even smaller so it agrees with $v \propto \frac{1}{\sqrt{R}}$ by multiplying by $\frac{\sqrt{R}}{\sqrt{R'}}$. Then we will have the true velocity represented by the long arrow.

Combining the two steps we see that in effect we are scaling the velocity arrow that is too long by applying the scaling factor:

$$\frac{R}{R^{'}} \times \frac{\sqrt{R}}{\sqrt{R^{'}}} = \left(\frac{R}{R^{'}}\right)^{\frac{3}{2}}$$

By inspection what we have done is decreased the actual velocity represented by the longer velocity arrow on the hodograph compared to what the longer velocity misleads us to believe is true. We used the adjustment $\left(\frac{R}{R^{'}}\right)^{\frac{3}{2}}$ to decrease the misleadingly long velocity arrow. This is equivalent mathematically to stating that the velocity is proportional to $\left(\frac{1}{R}\right)^{\frac{3}{2}}$ when comparing the velocities of different circular orbits around the same central body.

So when scaling circular hodographs the scaling rule is:

$$v \propto \left(\frac{1}{R}\right)^{\frac{3}{2}}$$

In other words we must correct the literal velocity when we plot two hodographs on the same diagram. The correct way to scale is to adjust the velocity arrows so that the velocity units of velocity arrows are proportional to $R^{\frac{3}{2}}$. For example, suppose that we plot two hodographs on the same diagram for planets in the same solar system and we have one planet that is 4 times farther away from the Sun.

If the units of scale for the smaller inner orbit are 100 kilometers per second, the units of scale for the larger orbit will be according to the rule for scaling:

$100 \times \frac{1}{4^{\frac{3}{2}}} = 100 \times \frac{1}{8} = 12.5$ kilometers per second.

So actually, even thought the velocity arrow for the larger orbit is four times larger and this would imply that the velocity would be 400 kilometers per second, we apply our scaling so that the unit of velocity on the larger arrow is 12.5 kilometers per second. The true velocity is, since the velocity arrow starts out before scaling to be literally four times longer than the small velocity arrow, $4 \times 12.5 = 50$ kilometers per second for the larger orbit. This is indeed correct since the true proportion for velocity is $v \propto \frac{1}{\sqrt{R}}$: In this case $v \propto \frac{1}{\sqrt{4}} \propto \frac{1}{2}$ and indeed 50 kilometers per second is one half of 100 kilometers per second.

So we now have a scaling method to use for comparing circular orbits of planets in the same solar system. We used the relation $T \propto R^{\frac{3}{2}}$ to teach us that we must scale the velocity arrow so that the velocity units of that arrow are proportional to $\frac{1}{R^{\frac{3}{2}}}$.