The Parts of an Ellipse

Orbits seem so simple. What could be so difficult about planets traveling their repetitive paths around the Sun? If they were indeed so easy to understand perhaps the challenge would be uninteresting. Perhaps it would take three pages instead of several hundred to explain. At a glance they seem to be no more complicated than a circle. But to err and overlook orbits as a manifestation of something deeper would be akin to contemplating a rainbow without noticing the colors. There are many aspects of orbits and each deserves its own little story. This is a rather long tale, told bit by bit.

It is best to start this book with a presentation of relevant aspects of the ellipse since each subsequent chapter will subject some property of the ellipse to productive scrutiny. The most thorough way to know a geometrical shape is to know its parts and their functions. The circle can be visualized as a radius, center, and curve. It is easy to visualize the radius spinning about the center to create the curve. As we seek to understand the ellipse, we should be aware that we are dealing with quite a different entity. The ellipse can vary in subtle ways despite conforming to a simple definition;

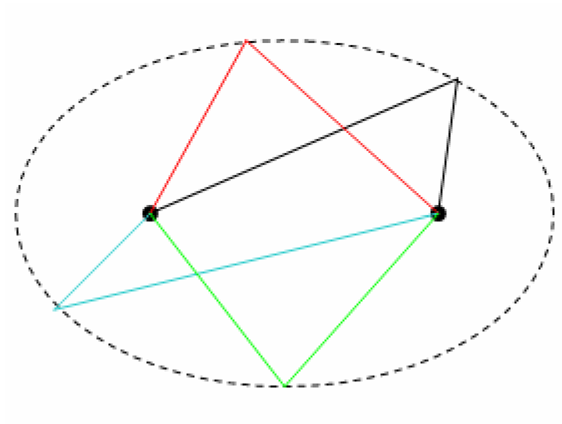
furthermore, there are many different ways to define an ellipse. Fortunately, we will mostly require only one.

The key parts of the ellipse for our purposes are the major axis, minor axis, focus, and semilatus rectum. These parts will be described in this chapter. In subsequent chapters, these parts will be shown to have functional roles related to the movements of planets.

The string and tack definition of an ellipse:

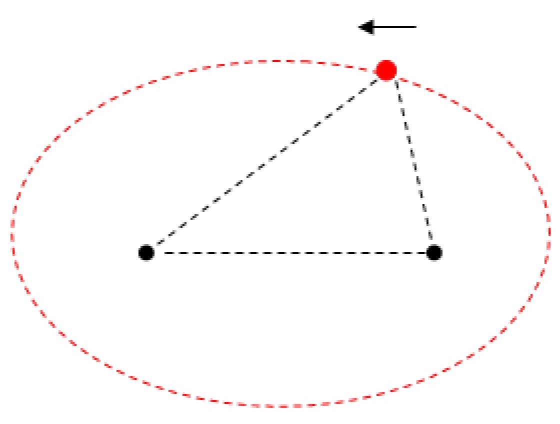
An ellipse is a carefully defined curve. It resembles an oval but must comply with a simple mathematical prescription. Start with two fixed points in space and draw a straight line, which we shall call a leg, leading away from each point. Orient the legs so that their ends meet:

Keep the sum of the length of the two legs constant. Under these conditions the tips of the legs always meet on a curve that is an ellipse:



Each color represents a set of legs in the ellipse above. The combined length of the two green legs is the same as the combined length of the two red legs or the two black legs. What is drawn in the ellipse above is the "string and tack" definition of the ellipse that will be so important in this book. Imagine that the two black points are tacks that are hammered into a board. A string of a definite length is cut and its ends are tied together so that the string now exists as a loop of string of constant length. The loop of string is placed around the tacks on the board and is pulled taut by a pencil to result in a curved shape pencilled onto the board.

In the figure below, the string of constant length is represented by the dashed black line. The tip of the pencil draws an ellipse represented by the red dashed line as the pencil pulls the string taut on its way around.

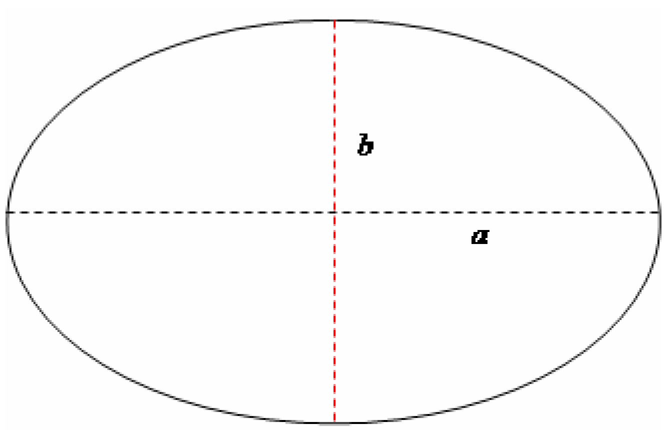


The string and tack ellipse is mathematically valid. It fulfills the strict mathematical definition of an ellipse. In a later chapter, specific mathematical equations related to the legs of the ellipse will be demonstrated.

The two tacks in the board correspond to the two fixed points in space. Each of the two points in space that is used to generate an ellipse is designated a "focus" of the ellipse. In later chapters it will be shown that the Sun is at one focus of the ellipse and the other focus is physically empty.

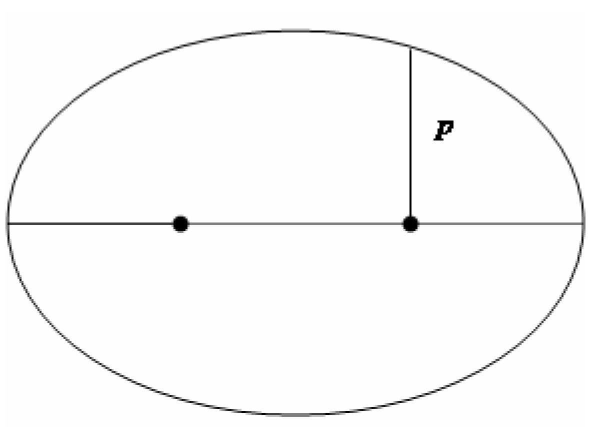
The major axes of an ellipse:

Several parts of the ellipse have names and descriptions by convention. Notice in figure below the two major axes of an ellipse. The dashed black line represents the long axis of the ellipse which is called the major axis. Half of the major axis, reaching from the center of the ellipse to the curve of the ellipse, is designated the semimajor axis and by convention is labeled with the letter a. The red dashed line represents the minor axis of the ellipse. Similarly, half of the minor axis is designated the semiminor axis and by convention is labeled with the letter b.



The semilatus rectum of an ellipse:

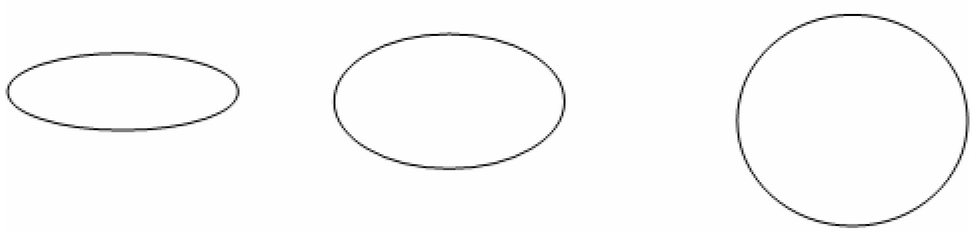
The line that is perpendicular to the major axis that begins at a focus and ends on the curve of the ellipse is called the semilatus rectum and is by convention designated with the letter p. Here we choose to draw the semilatus rectum on the right side of the ellipse. We could certainly have drawn one from the left focus. Arbitrarily, throughout most of the book, we assign the right focus of the ellipse to be where the Sun is located.



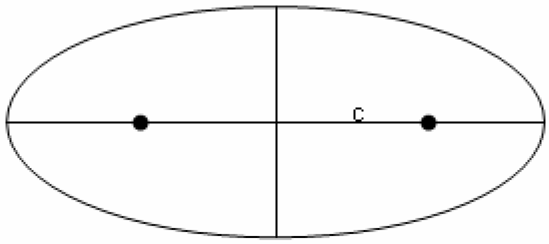
Throughout the book we usually concentrate on the semilatus rectum at the focus where the Sun is located. This semilatus rectum is referred to in *Orbits Explained* as the “Sunny side” semilatus rectum. We usually ignore the semilatus rectum at the other focus of the ellipse. As stated above, this other focus is an empty location in space, containing neither Sun nor planet.

Eccentricity of ellipses:

Ellipses vary in eccentricity. An ellipse that is flat and long is said to be highly eccentric. Many comets travel in long flat highly eccentric elliptical orbits. On the other hand the most round ellipse is actually a circle. A circle is an ellipse whose two foci coincide at the center of the circle. It is , however, definitely an ellipse since it fits the definition of an ellipse for which the two fixed points in space, the two foci, coincide. Many chapters in this book refer to the concept of a "family" of ellipses. An ellipse family is united by a common semimajor axis length. No matter how flat or round the ellipse is within a family, the length of the semimajor axis, a, is the same. The circle is a very important member of this family. It has a radius equal in length to the length of the semimajor axis of its elliptical counterparts. Of course, another way to state the same thing is to say that the length of the diameter of this circle is equal to the length of the major axis of all these members of an elliptical family.



Eccentricity, which is designated with the letter e, can be quantified. It is conventional to designate c to be the distance from the center of the ellipse to the focus. By inspection , it is evident that as the ellipse becomes flatter and more elongated, the foci become farther away from the center of the ellipse. So the length c becomes a greater proportion of the length a as the ellipse becomes more eccentric.



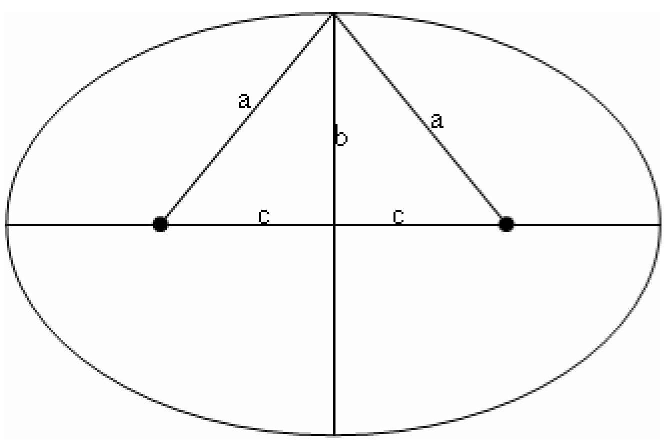
So, consistent with expectation, the eccentricity of an ellipse is defined mathematically as $\frac{c}{a}$.

Ellipse legs sum to $2a$:

An important property of an ellipse is that the length of the two legs of the ellipse sum to $2a$. This can best be seen when the two legs of the ellipse meet at one end of the major axis. By inspection of the ellipse above, one can see that the distance from one of the foci to the near end of the major axis is equal to $a-c$. The distance from the other focus to the same end of the major axis is $a+c$. These sum to $2a$ since $(a+c)+(a-c)=2a$.

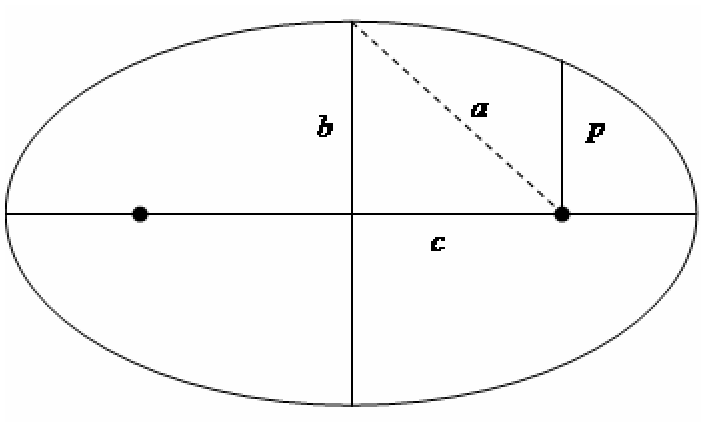
The relationship between a, b, and c:

There is an important relationship to note about the distance from a focus to the top of the semiminor axis. When the legs meet at the top of the semiminor axis two equal right triangles exist. The right angles in the two triangles are equal and so are two known sides, c and b. Thus, by the "side-angle-side" property of equal triangles, the third side, the hypotenuse of both right triangles, must be equal.. The sum of the length of the legs is equal to $2a$. Thus, each hypotenuse is equal to a. In other words, the distance from a focus to the top of the semiminor axis is equal to a .



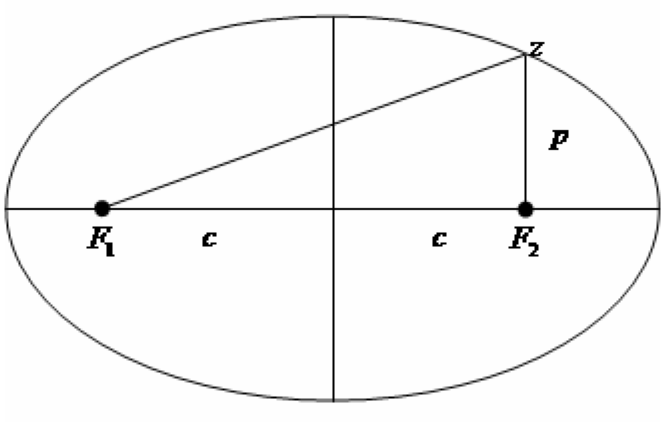
The equation of the semilatus rectum:

Notice that the above relationship results in a right triangle for which we can see that $b^2 + c^2 = a^2$. This is a formula that we can put to good use in order to define the semilatus rectum mathematically. The semilatus rectum is labeled p in the figure below.



Examine a different triangle in the figure below that uses the semilatus rectum as one of its sides. One of the legs of the ellipse is the semilatus rectum, p.

The other leg is the hypotenuse of a right triangle and is the segment $\overline{F_1 z}$. Note also that the base of the right triangle is equal to $2c$.



Since the sum of the two legs is equal to *2a* , the length of the segment $\overline{F_1 z}$ must be equal to *2a-p*.

The right triangle by the Pythagorean theorem demonstrates that:

$$(2c)^2 + p^2 = (2a - p)^2$$

Multiplying out:

$$4c^2 + p^2 = 4a^2 - 4ap + p^2$$

Cancel terms out:

$$4c^2 = 4a^2 - 4ap$$

Divide through by 4:

$$c^2 = a^2 - ap$$

Isolate p:

$$\frac{c^2 - a^2}{a} = -p$$

Multiply by minus one on both sides:

$$\frac{a^2 - c^2}{a} = p$$

Recall from earlier in this chapter that for an ellipse $b^2 + c^2 = a^2$, so that $a^2 - c^2 = b^2$. Substitute in the numerator:

$$\frac{b^2}{a} = p$$

So the semilatus rectum of an ellipse is related to the semiminor and semimajor axis of the ellipse.

A known expression that relates the semilatus rectum, p, to eccentricity is $p = a\left(1-e^2\right)$. It is not difficult to see that this is true using algebra. First restate $\left(1-e^2\right)$ in terms of a and b. Recall that $e = \frac{c}{a}$:

$$1-e^2 = 1-\frac{c^2}{a^2} = \frac{a^2}{a^2}-\frac{c^2}{a^2} = \frac{a^2-c^2}{a^2} = \frac{b^2}{a^2} \text{ since } b^2 = a^2-c^2$$

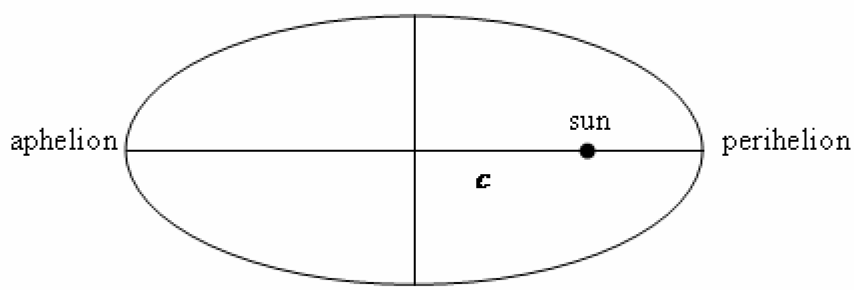
Now note that $\frac{b^2}{a^2}$ looks similar to the formula for p which is $\frac{b^2}{a}$ and compare the expression for p to the expression for $1-e^2$ to see that:

$$p = a\left(1-e^2\right).$$

<u>Perihelion and aphelion defined:</u>

This is a good place to digress and describe some aspects of the ellipse as it relates to the orbital path of

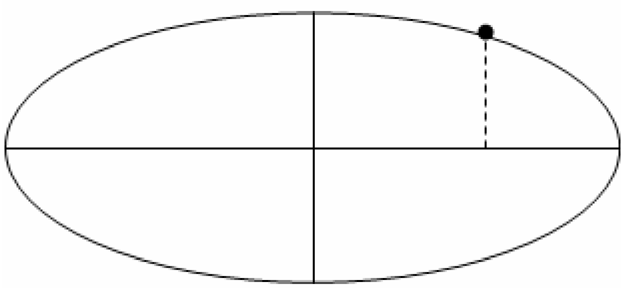
a planet as it orbits the Sun. The methods in subsequent chapters will show that the Sun is at one focus of the ellipse. The other focus remains empty as far as tangible things are concerned. The point along the major axis of the ellipse where the planet is closest to the Sun is called perihelion. By inspection of the figure we can see that the distance to perihelion from the Sun at one focus is $a-c$ which is equal to
$a-ae$ since $c = ae$ so that we can say the distance of perihelion is $a(1-e)$.



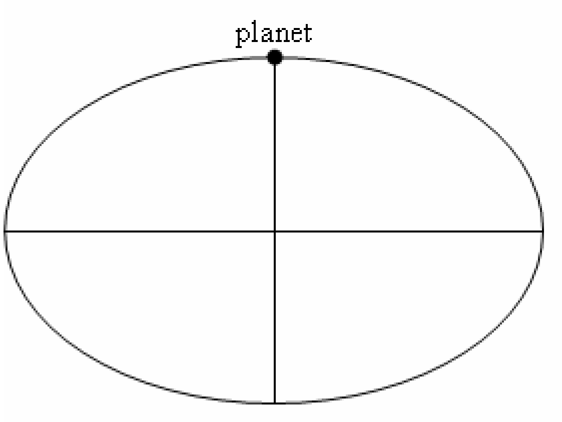
Using similar logic the distance to aphelion which is defined as the farthest point from the Sun along the major axis is $a+c$ which can be expressed as $a(1+e)$. As a double check we see that the sum of the perihelion distance and the aphelion distance is indeed 2a.

<u>Some positions on the curve of the ellipse</u>:

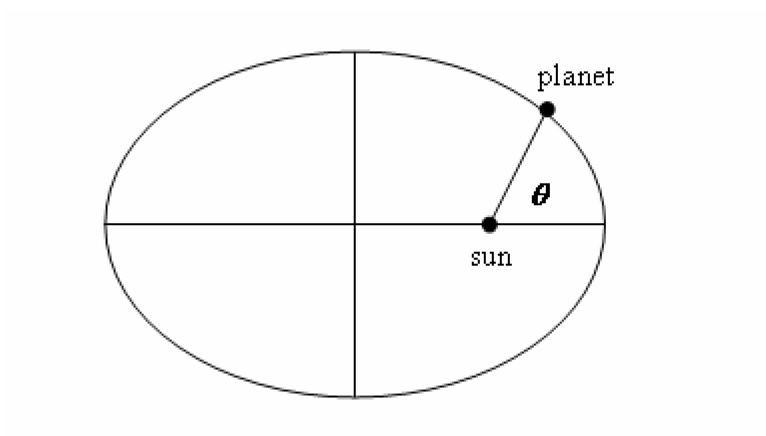
As a matter of clarification, at times in this book the planet is described as being at p. By this it is meant that the planet is at the end of the semilatus rectum.



Similarly when it is stated that the planet is at b, the planet is at the end of the seminor axis.

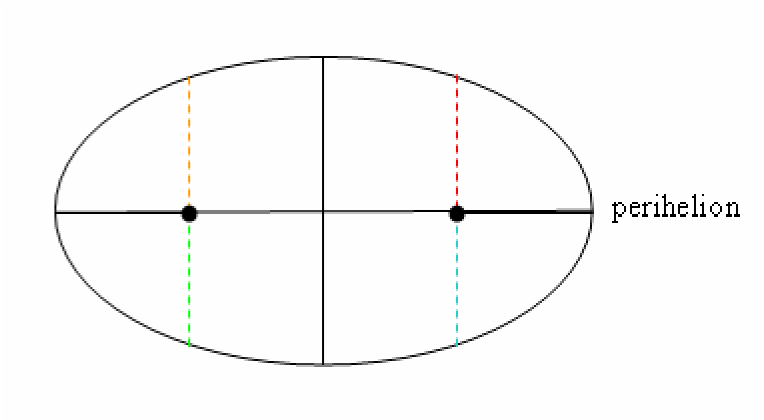


In the course of describing where the planet is, we speak of its angle relative to the semimajor axis on the side of perihelion, as measured from the focus containing the Sun. We often call this angle θ.



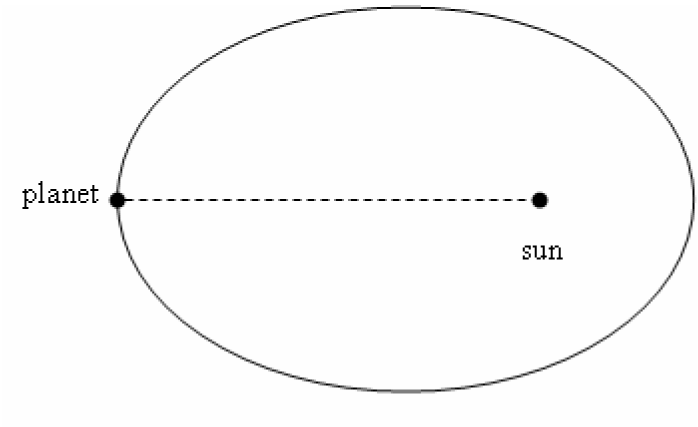
When the planet has completed an orbit , having started at perihelion we can say that θ is 360 degrees.

When the planet is at p the angle will be 90 degrees. In this book when we refer to the semilatus rectum we are usually referring to the one for which θ is 90 degrees. But notice that there are four possible semilatus recti in an ellipse.



We usually are referring to the semilatus rectum in red when discussing orbits in this book, with perihelion being on the right side of the above ellipse.

The planet is at 180 degrees when it is at aphelion.



So ends the brief description of some aspects of ellipse that need to be understood. With a common language it is easier to proceed.