Usiskin Logic

This chapter is a sidetrack into a philosophically engaging adventure. In Chapter 14 the representation of the total velocity in the hodograph was demonstrated. The tangent to the ellipse was taken to be the direction of motion. In this chapter the logic that determines the direction of motion. There is a beautiful chapter in Hogben that takes the reader on a philosophical journey. The reader is challenged to follow the stepwise philosophical and ancient proof by Plato that the square root of two is irrational. Inspired by the experience, in this chapter I will show that the tangent to an ellipse must be the direction of velocity for motion along the ellipse. We will give significant credit for the methods used here to Zalman P. Usiskin for describing a unique, in his own word, "pretrigonometric", proof of the reflective property of the tangent to the ellipse. In the spirit of his way of analyzing a problem, we need not adhere to a strict progression of mathematical equations. Instead, the compelling combination of logic and mere inspection will lead to a definitive proof that the tangent to the ellipse is along the true direction of motion.

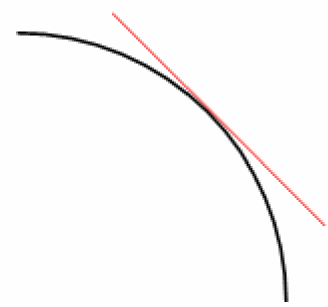
<u>Zalman P. Usiskin's Proof</u>

We know from Chapter 2 that the legs of the ellipse sum to a constant. We know from Chapter 14 that a ray of light emitted by one focus will reflect from the tangent line and travel to the second focus and that the angle of incidence is equal to the angle of reflection. We showed this using the Inverse Proportion Machine and mathematical methods. But Usiskin used logic to prove the reflective property of the ellipse. His reasoning is based on the fact that light travels from one specific point to another always via the shortest possible route, including when it is reflected off a reflective surface. The tangent line touches the ellipse at only one point. That point is where the legs of the ellipse meet. The length of the legs sum to $2a$. Anywhere else along the tangent line the lines from the two foci would sum to greater than $2a$ since anywhere else on the tangent line is outside the ellipse and anywhere outside the ellipse the legs sum to more than $2a$. So the shortest route from one focus to the other is along the legs of the ellipse. Since the shortest route is that taken by rays of light , it is proven that the tangent line and the legs of the ellipse comply with the reflective property of light. Inherent in the reflective property of light is the fact that the angle of incidence equals the angle of reflection. Hence the reflective property of the line tangent to an ellipse is proven.

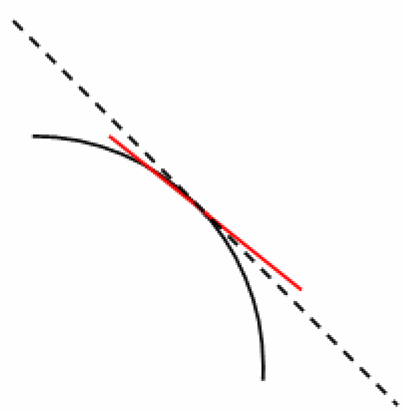
Logic gives the direction of total velocity

In admiration of Usiskin's methods I will show using logic that the line tangent to an ellipse represents the instantaneous direction of the velocity of the planet when it is at any position on the ellipse.

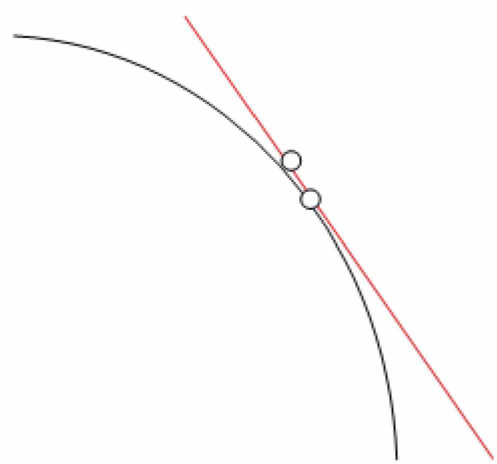
Adhere to what is accepted to be the ultimate definition of a tangent line to a curve. Visualize that the tangent line is the unique straight line that grazes but does not cross its curve and touches its curve at only one point.



How do we know that this is the only line that touches the curve only once? We could tilt the line slightly and feel that it still only touches once , if we do not look more carefully:

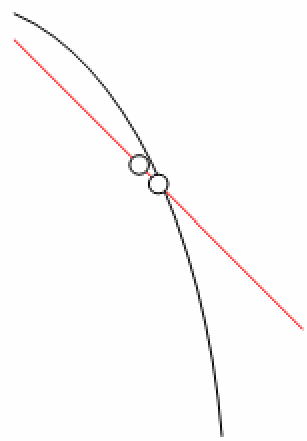


Let the red line satisfy the requirement that it touches the curve at only one point without crossing it. If we tilt it slightly clockwise it appears that the resultant dashed line still satisfies the requirement that it might only touch the curve once. But look closer. Magnify the curve and the straight line so greatly that we can see the point of contact for the original red tangent line and the very next point higher on the line.



Above , see that the red line touches the curve at only one point and that the next point is off the curve. But then tilt the line as slightly as imaginably is possible , still keeping the original point of contact touching the curve..

The very next point will at least touch the curve if it does not cross it:



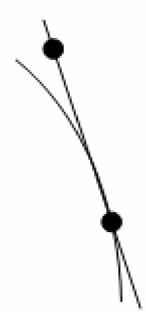
As seen above the original point of contact is still touching the curve. The very next point on the line has now crossed the curve.   Let's suppose that we decide to rotate the original tangent line even less, reasoning that we can finesse the situation and find a second line that only touches the curve once without crossing it.   If the very next point under those circumstances does not seem to cross the line, all that needs to be done is to imagine a point even closer to the original point of contact. The point that is even closer to the original point of contact will cross the curve if we place it close enough to the original point of contact.

What we have demonstrated is that for any curve there can be only one line that grazes it without crossing it that also touches the curve at only one point. Now examine the tangent line to an ellipse.

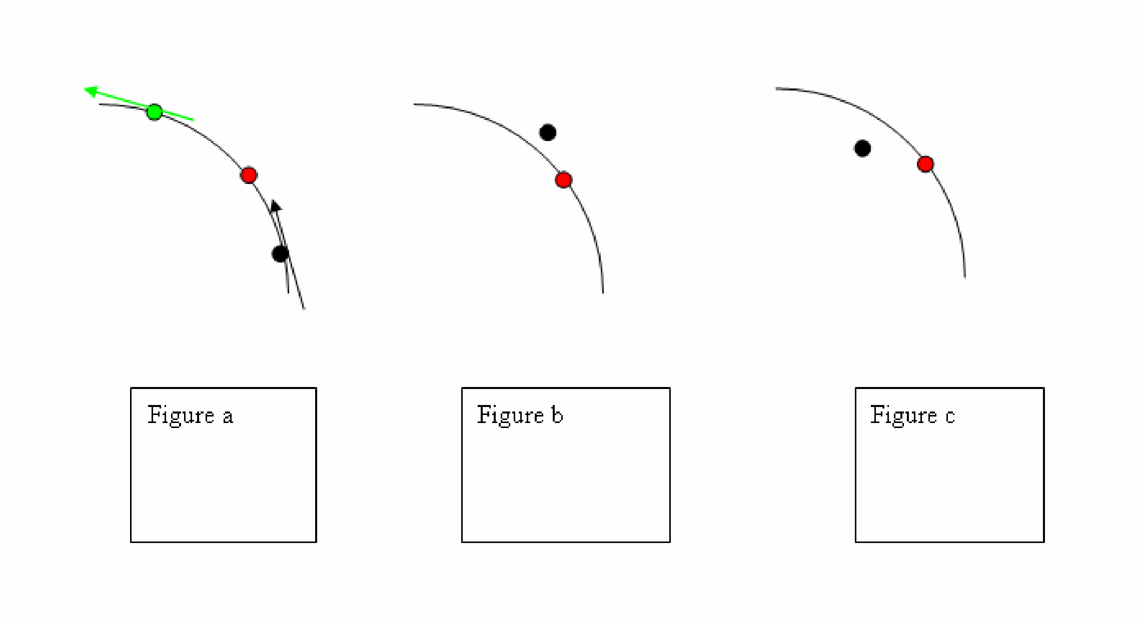
The tangent line to an ellipse must satisfy the requirements in order to be a tangent to a curve. It must touch the ellipse at only one point. The line that reflects the ray of light from one focus to the other touches the curve on the ellipse. The ellipse is defined by the curve that satisfies the requirement that the sum of the legs is a constant. Usiskin brilliantly observes that if we stay on this line and move to any other point aside from the point of contact, the sum of the legs of the ellipse will not be enough to reach it. So at any other point along this line, the line will be outside the curve of the ellipse. The line therefore touches the ellipse at only one point and satisfies the requirements to be the tangent line to the ellipse.

Examine the motion of the planet along the ellipse. The planet moves from one point on the ellipse to the very next point. Visualize these two adjacent points. Each point has its unique tangent line. And so does the next point and the next and so on. Visualize the act of drawing the ellipse with a pencil. As the pencil moves on the ellipse it continually changes direction just as a planet on the elliptical curve would. So if we can demonstrate the direction of motion of the pencil tip we can declare that direction to be valid for the planet. The pencil moves from one point to the very next. Each point has its unique tangent line.

Examine a point on a curve. It is also on the curve's tangent line. What can we say about its direction of motion? We can not yet say that it is traveling in the direction of the tangent line since that is what we are setting out to prove. But we can say that the direction of motion at the next point can not be on the same tangent line. If that were the case, the point would leave the curve:



In the previous figure the point is seen to leave the curve by the following logic; If its direction does not change, it must stay on the tangent line which only may touch the curve at one point. The point may not travel any distance at all along its tangent line or else it leaves the curve. The point thus must always leave one tangent line and move to the to the next.



In the figures above, the movement of the pencil tip along the ellipse is examined. The initial position is the black point on the curve. The tangent lines for the black and green points are indicated in their respective colors. An imaginary situation is set up so that the green point on the curve is the "almost adjacent" point at which the pencil tip will arrive. But we can always visualize an even closer adjacent point which is designated by the red point on the curve. This red point is the one that we will designate the true next position for the pencil tip - as close to the initial position of the pencil tip as is imaginable.

What is the direction of movement of the pencil tip when it is at the red point on the curve which is between the initial black and "almost adjacent" green points on the curve? The direction of movement can not be the same as the direction of the tangent line for the black point. If that were the case, then by inspection , the red point would move outside the curve as in figure *b*. The direction of movement can not be the same as the direction of the tangent line for the green point. If that were the case, the red point would wind up crossing the curve as in figure *c*. The direction of movement must be somewhere between the direction of the tangent line at the black point and the tangent line at the green point. We finally have a statement that relates the actual direction of motion to actual tangent lines.

All that remains is to imagine that the green, red, and black points become as close together on the curve as is imaginable. Examining points on a curve that are this close together means that we are examining such a small segment of the curve that the segment between the points begins to be virtually indistinguishable from a straight line. In that case the black and green tangent lines become virtually indistinguishable. Certainly, a tangent line at the red point would be indistinguishable from the other two tangent lines (black and green) since the red tangent line is between them and they are indistinguishable from each other. By the figures above, the direction of motion at the red point must be between the direction of the black and green tangent lines. It must be between the direction lines of two tangent lines (black and green) whose directions are virtually indistinguishable from each other. It must be in the direction of its own tangent line since the direction of its own tangent line is virtually indistinguishable from the direction of the other two (black and green). So the direction of motion for any point on an elliptical curve is in the direction of the tangent to the ellipse at that point.

We can not study total velocity of a planet without knowing its direction. In this chapter we have demonstrated how to find that direction.