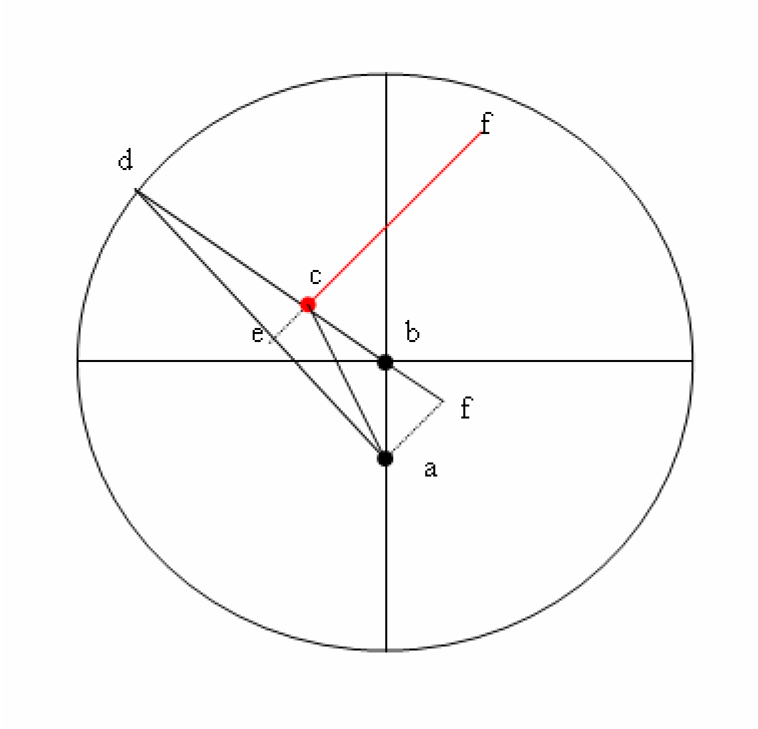
Total Velocity From the Hodograph

In Chapter 13 the visualization of the 90 degree rotation effect was explored. In this chapter the hodograph is examined in order to demonstrate the way in which the total velocity of the planet is represented. As a preliminary, the direction of the velocity will be indicated via a combination of logic and a property of ellipses that is known as the reflective property of the tangent to an ellipse.

<u>The reflective property of the tangent line to an ellipse</u>

There is a brilliant logical argument, regarding the direction of the total velocity arrows, presented in the book "Feynman's Lost Lecture" by David and Judith Goodstein. As any object moves along a curved path, its direction of motion at any instant is in the direction of the tangent to the curve. A tangent to a curve is a straight line that grazes the curve without crossing it. It logically is the direction that an object is moving if it is following the curve. Otherwise if it strays from the tangent line it will either cross the curve or fly away from it. Their book demonstrates a property of ellipses related to the two legs of the ellipse and the tangent to the point along the ellipse. This property is known as the reflective property of the ellipse which states that a ray of light leaving one focus will hit the tangent line at the ellipse and be reflected to the second focus. This property can be demonstrated by employing the knowledge that for rays of light hitting a mirrored surface, the angle of incidence is equal to the angle of reflection. We can show that this is true for the ellipse and our hodograph. We can do this by examining the arrangement of the legs of the ellipse and the velocity arrows generated by the Inverse Proportion Machine:



In the figure above generated by the Inverse Proportion Machine we saw in a previous chapter that the point e bisects the segment $\overline{da}$ and that segment $\overline{ec}$ is perpendicular to segment $\overline{da}$. By the side - angle - side rule of geometry, the triangles Δaec and Δdec are equal. Thus their angles $\angle ace$ and $\angle dce$ are equal to each other. The segment $\overline{cf}$ is an extension that we add to the hodograph temporarily now in order to demonstrate the angle $\angle fcb$. Note the intersecting lines $\overline{fe}$ and $\overline{bd}$. The rules of geometry tell us that opposite angles generated by intersecting lines are equal and so $\angle fcb = \angle dce$. Since $\angle dce$

also is equal to $\angle ace$, we can conclude that $\angle ace = \angle fcb$. So we have shown in the above hodograph that the legs of the ellipse $\overline{ac}$ and $\overline{bc}$ behave like rays of light reflecting off the line that represents the tangent to the ellipse $\overline{ef}$ since the angles of incidence and reflection, $\angle ace$ and $\angle fcb$, are equal.

What have we shown? We have shown that the legs of the ellipse generated by the hodograph satisfy the reflective property of the tangent to the ellipse and that the tangent of the ellipse is represented by the perpendicular line of the Inverse Proportion Machine - in this case the segment $\overline{ec}$. Now recall that we agreed upon a convention for hodographs that velocity arrows of a hodograph must be rotated by 90 degrees if the position of the radius is taken to be correct or, vice versa, that the directional orientation of the radius must be rotated by 90 degrees if the orientation of the velocity arrows is taken to be correct. In the hodograph above, we take the directional orientation of the radius and the position of the planet to be correct. It is therefore the velocity arrows that must be rotated by 90 degrees in order to be pointing in the correct direction. Note that by the Inverse Proportion Machine the segment $\overline{ec}$ is perpendicular to the segment $\overline{ad}$. Note that $\overline{ec}$ represents the tangent to the ellipse. It is the direction that the planet must be traveling since moving objects on a curve must travel on the tangent to the curve. We showed in a previous chapter that the direction of motion along an elliptical path is in the direction of the tangent to the curve. Note that if we rotate the segment $\overline{ad}$ by the requisite 90 degrees, so that it is parallel to segment $\overline{ec}$, it will be pointing in the correct direction for a velocity arrow - it is in the direction of the tangent to the ellipse. Segment $\overline{ad}$ does indeed qualify to represent total velocity in terms of direction . We must not forget to apply our convention to rotate either position or velocity by 90 degrees when we interpret a hodograph.

The shadow method determines total velocity

Stop and collect what we have so far. We have the magnitude and direction for the tangential velocity. We have the true direction for total velocity. Is this enough information to deduce the magnitude of the total velocity? Yes it is. Let the tangential velocity be represented in red. We know its magnitude and its direction relative to the total velocity since we know its direction relative to the tangent of the ellipse by our above discussion. For example we know the relative directions of the segments $\overline{ad}$ and $\overline{af}$ from their positions on the hodograph above. Segment $\overline{af}$ represents tangential velocity and segment $\overline{ad}$ represents total velocity. In any case, suppose the tangential velocity magnitude and direction are represented in red. For any velocity we may project its "shadow" onto the directional line we are interested in, so as to find the magnitude of velocity in that direction. The shadow method of analyzing velocity vector components was presented in a previous chapter. The shadow must be cast by light perpendicular to the direction of interest, meaning the direction of total velocity. The direction of the tangent to the ellipse is represented by the black dashed line. (Do not confuse the total velocity along the tangent line with tangential velocity. Tangential velocity is the component at a right angle to the radius from the planet to the Sun. Total velocity is in the direction of the tangent to the ellipse.) The tangential velocity is only a component of the total velocity. The tangential velocity acts like a cookie cutter. Rays from its ends parse the tangent direction (the direction of total velocity) into a segment of definite length. That length is the magnitude of the total velocity. So we have both the direction and magnitude of total velocity.